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Average pressure and velocity fields in non-uniform suspensions of spheres in Stokes flow

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Abstract. A widely used method for the approximate numerical simulation of the bulk behavior of particle suspensions consists in filling the entire space with copies of a fundamental cell in which N particles are arranged according to some probability distribution. Until now this method has only been used for suspensions that are spatially uniform in the mean. The case of spatially non-uniform systems, on the other hand, has not been considered. Here the average velocity and pressure fields for such a non-uniform suspension of identical rigid spheres in Stokes flow are calculated, and analytic solutions expressed in terms of multipole coefficients are presented. The results match and extend others obtained by the authors in parallel work using a completely different approach. In particular, the definition of a quantity to be identified with the mixture pressure is fully supported by the present results. An explicit result for the structure of the viscous stress in the suspension is also found. It is shown that, for spatially non-uniform systems, the stress contains a non-symmetric contribution analogous to a baroclinic source of vorticity.

As a byproduct of the analysis, certain integrals of two periodic functions introduced by Hasimoto are calculated. These integrals would arise in similar problems, *e.g.* the electric field produced by electric multipoles in a periodic cubic structure.

Key words: averaging, disperse flow, lattice sums, viscous suspensions.

1. Introduction

Suspensions of spheres in Stokes flow feature prominently among the most studied multiphase disperse systems (see, *e.g.*, [1–14]). In recent work in which direct numerical simulations have been carried out, a large volume of the suspension is usually approximated by filling the entire space with copies of a fundamental cell in which N identical rigid spheres are randomly arranged (see, *e.g.*, [15–18]). If the cell is large enough, the artificial periodicity induced by this construction may be expected to have a negligible effect and the system will approximate the bulk behavior of a suspension away from boundaries. The numerical results can be averaged and, if necessary for better statistics, ensembles of such systems can be generated, from which effective properties can be obtained.

Recently a method has been developed by which this approach – previously only applied to spheres arranged according to a uniform probability distribution – can be extended to spatially non-uniform probability distributions [16–20]. Non-uniform suspensions are, of course, of great interest as they give rise to average quantities that are not spatially uniform and therefore

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have non-zero spatial derivatives.[2] By using this fact and the techniques explained in [16–18], it is possible to find, in a systematic way, closure relations for averaged equations to describe the behavior of a suspension in a statistical sense.

It is these recent applications that motivate the present study in which ensemble averages of the Stokes pressure and velocity fields are explicitly evaluated. The results offer several elements of interest. In the first place, we find explicitly the structure of the stress tensor in the suspension, and prove the existence of an antisymmetric component even in the absence of external couples applied to the particles. This component can play an important role in the stability of the system. Secondly, by a completely different route, we recover the definition of mixture pressure proposed in [20] on the basis of a consideration of the transformation properties of the averaged equations upon a gauge transformation of the microscopic pressure. This finding strengthens the earlier derivation which, though somewhat unconventional, is applicable to finite Reynolds numbers and non-Newtonian fluids as well. Finally, unlike those of [16–18], the present results are exact to all orders in a/L, where a is the radius of the spheres and L the side of the cube, and, again unlike the earlier work, their derivation does not rely on a particular formulation of the average momentum equations for the phases.

As a final point of interest, we show how to evaluate volume integrals of general harmonic and biharmonic functions that are periodic in the cube and regular outside the spheres. The interest of this calculation lies in the fact that such integrals arise when these functions are expanded in a Fourier series in the cube; thus, the methods developed here are applicable beyond the specific application to Stokes flow that motivates their development.

2. Ensemble averages for a non-uniform system

Consider N identical spheres placed in a cubic volume $\mathcal{V}$ of side length L and immersed in a viscous liquid[3] (see Figure 1). We assume that the Reynolds number of the flow based on the sphere's radius a is so small that inertia is negligible and the Stokes equations apply. We also assume that the entire space is filled with copies of the cube so that the disturbance fields due to the presence of the spheres have the same periodicity as the cell structure.

Given a deterministic forcing agent, such as a force applied to the particles or an imposed shear, at every instant, the behavior of the system is entirely determined by the position of the sphere centers $\mathbf{y}^\alpha$, $\alpha = 1, 2, \ldots, N$. Let $P(N) \equiv P(\mathbf{y}^1, \mathbf{y}^2, \ldots \mathbf{y}^N)$ denote the probability density of the distribution of centers, normalized so that

$$\int \mathrm{d}\mathcal{C}^N\, P(N) = N!\,, \tag{2.1}$$

in view of the spheres being identical. Here $\mathrm{d}\mathcal{C}^N = \mathrm{d}^3y^1\, \mathrm{d}^3y^2\, \ldots \mathrm{d}^3y^N$, and, for each variable, the integration ranges over the entire volume $\mathcal{V}$. In general, P will also depend on time, but the dependence on this variable is non-essential and is omitted throughout the present paper

For each configuration $\mathcal{C}^N$, we introduce an indicator function for the volume occupied by the spheres:

[2] It is remarkable that, in spite of the large number of papers devoted to disperse multiphase systems, only very few explicitly address the spatially non-uniform situation [21–23].

[3] If the Fourier expansions that follow are generalized to include wave numbers of the reciprocal lattice, the present results are also applicable to a fundamental cell in the shape of a parallelepiped with slanted sides.

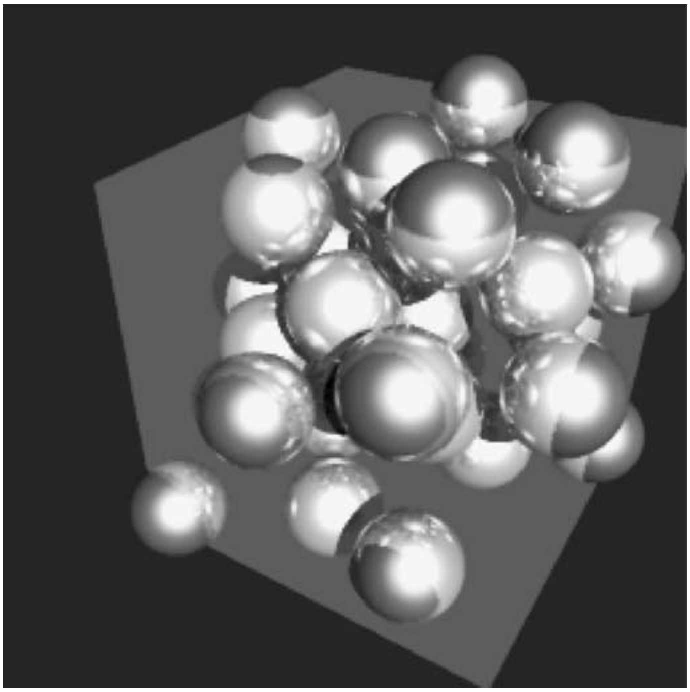

Figure 1. The system studied in this paper consists of a fundamental cell containing N spheres infinitely repeated so as to fill the entire space. This construction is widely used in the literature for the numerical simulation of macroscopically large (ideally infinite) volumes of a liquid-sphere suspension. For the example shown here the sphere volume fraction is 30%.

$$\chi(\mathbf{x}; N) = \sum_{\alpha=1}^{N} H\left(a - |\mathbf{x} - \mathbf{y}^{\alpha}|\right) , \tag{2.2}$$

where H is the Heaviside distribution, and a the common radius of the spheres. Clearly $\chi(\mathbf{x}; N) = 1$ if $\mathbf{x}$ is inside a sphere while $\chi(\mathbf{x}; N) = 0$ otherwise. The liquid volume fraction $\beta_L(\mathbf{x})$, defined as the probability that the point $\mathbf{x}$ is in the liquid phase, is given by

$$\beta_L(\mathbf{x}) = \frac{1}{N!} \int d\mathcal{C}^N\, P(N) \left[1 - \chi(\mathbf{x}; N)\right] , \tag{2.3}$$

and the phase-ensemble average for the generic liquid field f (such as pressure, velocity, etc.) is

$$\beta_L(\mathbf{x}) \langle f \rangle(\mathbf{x}) = \frac{1}{N!} \int d\mathcal{C}^N\, P(N)\, (1 - \chi)\, f(\mathbf{x}; N) . \tag{2.4}$$

If the field f is spatially periodic, all average quantities can be expanded in a Fourier series:

$$\beta_L(\mathbf{x})\langle f \rangle(\mathbf{x}) = f_0 + \sum_{\mathbf{k}\neq 0} f_{\mathbf{k}} \exp\left(-i\mathbf{k} \cdot \mathbf{x}\right) , \tag{2.5}$$

where the summation is over all wavenumbers that are compatible with the dimensions of the cell (excluding $\mathbf{k} = 0$), and

$$f_0 = \frac{1}{\mathcal{V}} \int d^3x\, \beta_L(\mathbf{x})\langle f \rangle(\mathbf{x}) , \qquad f_{\mathbf{k}} = \frac{1}{\mathcal{V}} \int d^3x\, \exp\left(i\mathbf{k} \cdot \mathbf{x}\right)\, \beta_L(\mathbf{x})\langle f \rangle(\mathbf{x}) , \tag{2.6}$$

or, from (2.4),

$$f_0 = \frac{1}{N!}\int \mathrm{d}\mathcal{C}^N\, P(N)\, \left[\frac{1}{\mathcal{V}}\int \mathrm{d}^3x\, (1-\chi)\, f(\mathbf{x};N)\right], \tag{2.7}$$

$$f_{\mathbf{k}} = \frac{1}{N!}\int \mathrm{d}\mathcal{C}^N\, P(N)\, \left[\frac{1}{\mathcal{V}}\int \mathrm{d}^3x\, (1-\chi)\, \exp\,(\mathrm{i}\mathbf{k}\cdot\mathbf{x})\, f(\mathbf{x};N)\right]. \tag{2.8}$$

The presence of the liquid characteristic function $(1-\chi)$ restricts the spatial integration domain in (2.7) and (2.8) to the liquid region $\mathcal{L}$ consisting of the fundamental cell minus the volumes σ^α of the spheres:

$$\mathcal{L} = \mathcal{V} - \bigcup_{\alpha=1}^{N}\sigma^\alpha . \tag{2.9}$$

Then, in order to calculate the Fourier coefficients f_0, $f_{\mathbf{k}}$, we need to calculate integrals of the form

$$F_0(N) = \int_{\mathcal{L}} \mathrm{d}^3x\, f(\mathbf{x};N), \tag{2.10}$$

$$F_{\mathbf{k}}(N) = \int_{\mathcal{L}} \mathrm{d}^3x\, \exp\,(\mathrm{i}\mathbf{k}\cdot\mathbf{x})\, f(\mathbf{x};N). \tag{2.11}$$

3. Particle averages

It will be shown in the following sections that, by use of Green's identity, the volume integrals (2.10) and (2.11) that arise in the present problem can be reduced to sums of the type

$$F_0(N) = \sum_{\alpha=1}^{N} J_0^\alpha \qquad F_{\mathbf{k}}(N) = \frac{1}{k^{2M}}\sum_{\alpha=1}^{N}\exp\,(\mathrm{i}\mathbf{k}\cdot\mathbf{y}^\alpha)\, I_{\mathbf{k}}^\alpha , \tag{3.1}$$

where J_0^α, $I_{\mathbf{k}}^\alpha$ are integrals over the surface of the α-th particle to be defined later, M is a non-negative integer, and $k = |\mathbf{k}|$; the Fourier coefficients (2.7) and (2.8) then take the form

$$f_0 = \frac{1}{N!}\int \mathrm{d}\mathcal{C}^N P(N)\frac{1}{\mathcal{V}}\sum_{\alpha=1}^{N} J_0^\alpha , \tag{3.2}$$

$$f_{\mathbf{k}} = \frac{1}{N!}\int \mathrm{d}\mathcal{C}^N P(N)\frac{1}{\mathcal{V}}\frac{1}{k^{2M}}\sum_{\alpha=1}^{N}\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{y}^\alpha)\, I_{\mathbf{k}}^\alpha . \tag{3.3}$$

In order to write the results that follow in a more compact form, it is useful to introduce a different kind of average that we refer to as *particle average*.

Consider a generic quantity g^α pertaining to each particle α as a whole, such as the center-of-mass velocity, J_0^α, $I_{\mathbf{k}}^\alpha$, etc. We define the particle average of g^α, denoted by an overbar, according to

$$n(\mathbf{x})\overline{g}(\mathbf{x}) = \frac{1}{N!}\int \mathrm{d}\mathcal{C}^N P(N)\left[\sum_{\alpha=1}^{N}\delta(\mathbf{x}-\mathbf{y}^\alpha)g^\alpha(N)\right], \tag{3.4}$$

where

$$n(\mathbf{x}) = \frac{1}{N!}\int \mathrm{d}\mathcal{C}^N P(N)\sum_{\alpha=1}^{N}\delta(\mathbf{x}-\mathbf{y}^\alpha)\ , \tag{3.5}$$

is the local particle number density. In the present spatially periodic system, particle averages can also be expanded in a Fourier series:

$$n(\mathbf{x})\overline{g}(\mathbf{x}) = (ng)_0 + \sum_{\mathbf{k}\neq 0}(ng)_{\mathbf{k}}\exp\left(-\mathrm{i}\mathbf{k}\cdot\mathbf{x}\right)\ , \tag{3.6}$$

with

$$(ng)_0 = \frac{1}{\mathcal{V}}\int \mathrm{d}^3x\, n(\mathbf{x})\overline{g}(\mathbf{x})\ , \qquad (ng)_{\mathbf{k}} = \frac{1}{\mathcal{V}}\int \mathrm{d}^3x\ \exp\left(\mathrm{i}\mathbf{k}\cdot\mathbf{x}\right) n(\mathbf{x})\overline{g}(\mathbf{x})\ , \tag{3.7}$$

or, from (3.4),

$$(ng)_0 = \frac{1}{N!}\int \mathrm{d}\mathcal{C}^N P(N)\frac{1}{\mathcal{V}}\sum_{\alpha=1}^{N} g^\alpha\ , \tag{3.8}$$

$$(ng)_{\mathbf{k}} = \frac{1}{N!}\int \mathrm{d}\mathcal{C}^N P(N)\frac{1}{\mathcal{V}}\sum_{\alpha=1}^{N}\exp\left(\mathrm{i}\mathbf{k}\cdot\mathbf{y}^\alpha\right) g^\alpha\ . \tag{3.9}$$

Upon comparing (3.2) with (3.8), we observe that f_0 is just the $\mathbf{k} = 0$ Fourier coefficient of n times the particle average of J^α, $f_0 = (nJ)_0$ and, from (3.3) and (3.9), we similarly see that $k^{2M} f_{\mathbf{k}} = (nI)_{\mathbf{k}}$ so that, from (2.5),

$$\begin{aligned}\beta_L\langle f\rangle &= (nJ)_0 + \sum_{\mathbf{k}\neq 0} k^{-2M}\,(nI)_{\mathbf{k}}\exp\left(-\mathrm{i}\mathbf{k}\cdot\mathbf{x}\right)\\ &= (nJ)_0 + \left(-\nabla^2\right)^{-M}\sum_{\mathbf{k}\neq 0}(nI)_{\mathbf{k}}\exp\left(-\mathrm{i}\mathbf{k}\cdot\mathbf{x}\right)\ ,\end{aligned} \tag{3.10}$$

where $(\nabla^2)^{-1}$ is the formal inverse of the Laplacian in the space of functions periodic in the cube. Noting that, according to the expansion (3.6),

$$n(\mathbf{x})\overline{I}(\mathbf{x}) = (nI)_0 + \sum_{\mathbf{k}\neq 0}(nI)_{\mathbf{k}}\exp\left(-\mathrm{i}\mathbf{k}\cdot\mathbf{x}\right)\ , \tag{3.11}$$

we then have

$$\beta_L\langle f\rangle = \left(n\overline{J}\right)_0 + (-\nabla^2)^{-M}\left[n\overline{I} - \left(n\overline{I}\right)_0\right]\ . \tag{3.1}$$

These formulae are useful to simplify the expression of the results that follow.

4. Velocity and pressure fields in a periodic suspension

Consider a fundamental cell consisting of a liquid-filled cube of side length L in which N identical spherical rigid particles of radius a are randomly located (Fig. 1). We will solve the fluid mechanical problem of interest in this cell subject to periodicity boundary conditions at the cell surfaces, a procedure which may equivalently be viewed as solving the problem in the entire space filled with an infinite repetition of this cell. If the fundamental cell is large enough, it may be expected that, in spite of the artificial spatial periodicity of the construction, this arrangement would approximate the behavior of a macroscopic suspension away from macroscopic boundaries and large enough to contain many particles.[4] In this arrangement, the centers of the spheres belong to a system of N infinite interpenetrating periodic lattices.

Each particle is subjected to an external force $\mathbf{F}^\alpha$, has a translational velocity $\mathbf{w}^\alpha$, and an angular velocity $\boldsymbol{\Omega}^\alpha$, with $\alpha = 1, 2, \ldots, N$. We further assume that the Reynolds number for the relative particle-fluid motion is so small that the Stokes flow equations apply and that the particle inertia is also negligible. Because of this last assumption, the imposed external force $\mathbf{F}^\alpha$ must be exactly balanced by the hydrodynamic force on the particle.

As in Mo and Sangani [8] we write the liquid velocity field as

$$\mathbf{u}(\mathbf{x}) = \mathbf{U}_\infty(\mathbf{x}) + \mathbf{v}(\mathbf{x}) \;, \qquad \mathbf{v}(\mathbf{x}) = \sum_{\alpha=1}^{N} \mathbf{u}^\alpha(\mathbf{x}) \;, \tag{4.1}$$

where $\mathbf{U}_\infty$ is an imposed deterministic velocity field and $\mathbf{u}^\alpha$ is the disturbance induced by the α-th particle; for simplicity, we only consider a uniform or linear imposed velocity:

$$\mathbf{U}_\infty = \mathbf{U}_0 + \boldsymbol{\gamma} \cdot \mathbf{x} \;; \tag{4.2}$$

a note at the end of this section will indicate how more general forms for $\mathbf{U}_\infty$ can be accounted for. Since we only consider a situation in which inertia is unimportant, by a suitable choice of the frame of reference we can eliminate the solid-body rotation corresponding to the anti-symmetric part of the constant tensor $\boldsymbol{\gamma}$ which will, therefore, be assumed symmetric in the following; incompressibility also requires $\boldsymbol{\gamma}$ to be traceless. By a similar argument we could also take $\mathbf{U}_0 = 0$ but, for the sake of the transparency of some of the relations that follow, we will not do so. The velocity $\mathbf{U}_\infty$ defined in (4.2) diverges at infinity; this is an artifact of the assumed infinite extent of the system with no physical consequences provided the particle velocities are also adjusted in such a way that the disturbance velocity, $\mathbf{u} - \mathbf{U}_\infty$, is finite and the same in each cell. Any conceptual difficulty may be avoided by focusing on the fundamental cell in which the problem is solved subject to periodicity boundary conditions.

In their Equation (28) Mo and Sangani [8] give an expression for the pressure disturbance corresponding to $\mathbf{u}^\alpha$; after using their Equations (31) to (33) and (70), this may be written as

$$p^\alpha(\mathbf{x}) = \frac{1}{\mathcal{V}}\mathbf{x} \cdot \mathbf{F}^\alpha - \mu \mathcal{G}^\alpha \cdot \nabla S_1\left(\mathbf{x} - \mathbf{y}^\alpha\right) \;, \tag{4.3}$$

where μ is the liquid viscosity and the symbol $\mathcal{G}^\alpha$ denotes a differential operator defined in [8] that we do not need to write down explicitly. The function S_1, a modified Green's function introduced by Hasimoto [28], satisfies

$$\nabla^2 S_1(\mathbf{x}) = 4\pi \left[\frac{1}{\mathcal{V}} - \sum_{\mathbf{x}_K} \delta(\mathbf{x} - \mathbf{x}_K) \right] \,, \tag{4.4}$$

[4] The use of a fundamental cell in which the problem of interest is solved subject to periodicity boundary conditions is a standard device in molecular dynamics (see, *e.g.*, [24, p. 92], and [25]), granular flow (see, *e.g.*, [26]), composites (see, *e.g.*, [15]), suspensions (see, *e.g.*, [5, 27, 8]), and others.

where the poles of the delta distribution are located at the centers of an infinite system of cubic cells of side L; the summation is extended to all the cell centers $\mathbf{x}_K$ including the origin. This relation shows that $S_1(\mathbf{x}-\mathbf{y}^\alpha)$ is singular at the center $\mathbf{y}^\alpha$ of the α-th particle.

We write the total pressure in the form

$$p(\mathbf{x}) = P_\infty(\mathbf{x}) + \mu q(\mathbf{x}) \,, \tag{4.5}$$

where, as suggested by (4.5),

$$q(\mathbf{x}) = -\sum_{\alpha=1}^{N} \mathcal{G}^\alpha \cdot \nabla S_1(\mathbf{x}-\mathbf{y}^\alpha) \,; \tag{4.6}$$

by construction q and $\mathbf{v}$ satisfy the Stokes flow equation:

$$-\boldsymbol{\nabla} q + \nabla^2 \mathbf{v} = 0 \,. \tag{4.7}$$

In order to find the field P_∞ we substitute p and $\mathbf{u}$ in the Stokes equation

$$-\boldsymbol{\nabla} p + \mu \nabla^2 \mathbf{u} = -\mathbf{G} \,, \tag{4.8}$$

where $\mathbf{G}$ is the body force per unit volume acting on the continuous phase (*e.g.*, $\mathbf{G} = \rho\mathbf{g}$, with ρ the continuous phase density, in the case of gravity); by use of (4.7) we then find, up to a constant,

$$P_\infty = \mathbf{x} \cdot \left(\frac{1}{\mathcal{V}} \sum_{\alpha=1}^{N} \mathbf{F}^\alpha + \mathbf{G} \right) \,, \tag{4.9}$$

which agrees with Equation (73) of [8] given that, here, $\mathbf{F}^\alpha$ is the imposed force on the particle which is exactly balanced by the hydrodynamic force. As will be clear from Section 5.3, this relation simply states that the static pressure gradient in a suspension exceeds that in the pure suspending phase by the contribution of the forces acting on the particles.

In the following, we shall rely on local expansions of q and $\mathbf{v}$ in the neighborhood of the α-th particle. For this purpose, we use the general solution of the Stokes flow equations given by Lamb [29, article 336] (see also [30, pp. 83–93]) and write

$$q(\mathbf{y}^\alpha + \mathbf{r}) = \sum_{-\infty}^{\infty} q_n^\alpha(\mathbf{r}) \,, \tag{4.10}$$

$$\begin{aligned} \mathbf{v}(\mathbf{y}^\alpha + \mathbf{r}) = \sum_{n=-\infty}^{\infty} \Bigg[& \frac{n+3}{2(n+1)(2n+3)} r^2 \boldsymbol{\nabla} q_n^\alpha - \frac{n}{(n+1)(2n+3)} \mathbf{r} q_n^\alpha \\ & + \boldsymbol{\nabla} \times (\mathbf{r}\chi_n^\alpha) + \boldsymbol{\nabla}\phi_n^\alpha \Bigg] \,, \end{aligned} \tag{4.11}$$

where each one of the potentials $q_n^\alpha, \phi_n^\alpha, \chi_n^\alpha$ is a solid harmonic function of order n. For example, for $n \geq 0$, the q_n^α's are regular at $\mathbf{r} = 0$ and are given by

$$q_n^\alpha(\mathbf{r}) = \left(\frac{r}{a}\right)^n \sum_{m=0}^{n} P_n^m(\cos\theta) \left[q_{nm}^{r\alpha} \cos(m\varphi) + \tilde{q}_{nm}^{r\alpha} \sin(m\varphi) \right] \,, \tag{4.12}$$

while harmonics with a negative index are singular and are given by

$$q^{\alpha}_{-n-1}(\mathbf{r}) = \left(\frac{a}{r}\right)^{n+1} \sum_{m=0}^{n} P_n^m(\cos\theta)\left[q^{\alpha}_{nm}\cos(m\varphi) + \tilde{q}^{\alpha}_{nm}\ \sin(m\varphi)\right] . \tag{4.13}$$

Here (r, θ, φ) are spherical coordinates centered at $\mathbf{y}^{\alpha}$ and the P_n^m's are associated Legendre functions. For $n = -1$ the first two terms of (4.11) cancel each other and ϕ^{α}_{-1} gives the velocity field of a simple source. For spheres of constant volume, as those considered here, one must therefore have $\phi^{\alpha}_{-1} = 0$.

The no-slip condition at the particle surface requires that

$$\mathbf{v}(\mathbf{y}^{\alpha} + \mathbf{r}) + \mathbf{U}_{\infty}(\mathbf{y}^{\alpha} + \mathbf{r}) = \mathbf{w}^{\alpha} + \boldsymbol{\Omega}^{\alpha} \times \mathbf{r} , \tag{4.14}$$

where $\boldsymbol{\Omega}^{\alpha}$ is the angular velocity of particle α (with respect to that of the frame of reference, *cf.* comment following Equation (4.2)); thus

$$\mathbf{v}(\mathbf{y}^{\alpha} + \mathbf{r}) = \mathbf{w}^{\alpha}_0 + \boldsymbol{\Omega}^{\alpha} \times \mathbf{r} - \boldsymbol{\gamma} \cdot \mathbf{r} , \tag{4.15}$$

with

$$\mathbf{w}^{\alpha}_0 = \mathbf{w}^{\alpha} - \mathbf{U}_0 - \boldsymbol{\gamma} \cdot \mathbf{y}^{\alpha} . \tag{4.16}$$

The periodicity of the problem requires that $\mathbf{w}^{\alpha}_0$ be equal for the corresponding particles in each cell, which makes $\mathbf{w}^{\alpha}$ larger and larger in the more distant cells; again this is an inconsequential artifact deriving from the infinite extent of the system and the assumed periodicity of the disturbance flow induced by the particles.

By using (4.15) we can readily show that the following relations between the singular and regular potentials hold:

$$q^{\alpha}_n = \frac{(n+1)(2n+3)}{n}\left(\frac{r}{a}\right)^{2n+1}\left[\frac{1}{2}q^{\alpha}_{-n-1} - \frac{2n+1}{a^2}\phi^{\alpha}_{-n-1}\right] , \tag{4.17}$$

$$\begin{aligned} \phi^{\alpha}_n = {} & \frac{n+1}{2n}\left(\frac{r}{a}\right)^{2n+1}\left[(2n+3)\phi^{\alpha}_{-n-1} - \frac{2n+1}{2(2n-1)}a^2 q^{\alpha}_{-n-1}\right] \\ & + \mathbf{w}^{\alpha}_0 \cdot \mathbf{r}\delta_{n1} - \frac{1}{2}\mathbf{r} \cdot \boldsymbol{\gamma} \cdot \mathbf{r}\,\delta_{n2} , \end{aligned} \tag{4.18}$$

$$\chi^{\alpha}_n = -\left(\frac{r}{a}\right)^{2n+1}\chi^{\alpha}_{-n-1} + \boldsymbol{\Omega}^{\alpha} \cdot \mathbf{r}\,\delta_{n1} . \tag{4.19}$$

It is an immediate consequence of these relations that $q^{\alpha}_{-1} = \phi^{\alpha}_{-1} = 0$, as expected, as particles of constant volume cannot give a monopole contribution.

For an imposed flow more general than the linear one (4.2) considered here, these relations would contain additional terms analogous in character to the last two in (4.18).

5. Fourier expansion of the pressure field

It is the objective of this section to obtain a Fourier representation of the form (2.5) for the liquid pressure disturbance q. Since in this calculation we only use the fact that q is harmonic, the results hold for an arbitrary periodic harmonic function.

We begin by considering integrals of the type (2.11) for non-zero wavenumbers $\mathbf{k}$; the method for the integral (2.10) then readily follows and is given later.

5.1. Non-zero wave number

The integral (2.11) for q is

$$Q_{\mathbf{k}} = \int_{\mathcal{L}} \mathrm{d}^3x \exp\left(\mathrm{i}\mathbf{k}\cdot\mathbf{x}\right) q(\mathbf{x}) \ , \tag{5.1}$$

or, given that $\nabla^2 \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x}) = -k^2 \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x})$,

$$Q_{\mathbf{k}} = -\frac{1}{k^2}\int_{\mathcal{L}} \mathrm{d}^3x \left[\nabla^2 \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x})\right] q(\mathbf{x}) \ . \tag{5.2}$$

The expression (4.6) for q and the relation (4.4) satisfied by S_1 show that q is harmonic away from the particle centers, which do not belong to the integration domain $\mathcal{L}$. Hence, after an application of Green's identity, we are left with an integral over the total boundary A of the integration domain consisting of the surfaces of the N spheres and of the surface of the fundamental cell:

$$Q_{\mathbf{k}} = \frac{1}{k^2}\int_{A} \mathrm{d}A \left[q(\mathbf{x})\boldsymbol{\nabla} \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x}) - \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x})\boldsymbol{\nabla} q(\mathbf{x})\right]\cdot\mathbf{n} \ , \tag{5.3}$$

where the unit normal $\mathbf{n}$ points into the fluid. The contribution of the latter surface vanishes by periodicity, and the end-result is an integral over the surfaces of the N spheres as in (3.1):

$$Q_{\mathbf{k}} = \sum_{\alpha=1}^{N} \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{y}^{\alpha}) L^{\alpha}_{\mathbf{k}} \ , \tag{5.4}$$

where

$$L^{\alpha}_{\mathbf{k}} = \frac{1}{k^2}\int_{r=a} \mathrm{d}S^{\alpha}\mathbf{n}\cdot\left[q(\mathbf{y}^{\alpha}+\mathbf{r})\boldsymbol{\nabla} \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}) - \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})\boldsymbol{\nabla} q(\mathbf{y}^{\alpha}+\mathbf{r})\right] \ , \tag{5.5}$$

and differentiation is with respect to the variable $\mathbf{r}$.

By use of the formula

$$\mathbf{r}\cdot\boldsymbol{\nabla} q^{\alpha}_{n}(\mathbf{r}) = n q^{\alpha}_{n}(\mathbf{r}) \ , \tag{5.6}$$

which immediately follows from (4.12) and (4.13), we can readily show that

$$L^{\alpha}_{\mathbf{k}} = \sum_{n=-\infty}^{\infty} \frac{1}{a}\left(k\frac{\partial}{\partial k} - n\right)\int_{r=a} \mathrm{d}S^{\alpha} \ \exp(\mathrm{i}ka \ \cos\theta) q^{\alpha}_{n}(\mathbf{r}) \ , \tag{5.7}$$

where $k = |\mathbf{k}|$ and the angle θ is measured from the direction of $\mathbf{k}$.

In view of the well-known relation

$$\exp(\mathrm{i}ka \ \cos\theta) = \sum_{n=0}^{\infty} \mathrm{i}^n (2n+1) j_n(ka) P_n(\cos\theta) \ , \tag{5.8}$$

and of the orthogonality of spherical harmonics, (5.7) becomes

$$L^{\alpha}_{\mathbf{k}} = \frac{3v}{ka}\sum_{l=0}^{\infty} \mathrm{i}^l \left[\frac{2l+1}{ka} j_l(ka) q^{\alpha}_{l0} - \left(q^{r\alpha}_{l0} + q^{\alpha}_{l0}\right) j_{l+1}(ka)\right] \ . \tag{5.9}$$

Here $v = \frac{4}{3}\pi a^3$ is the particle volume and it may be recalled that $q_{n0}^{r\alpha}$ is a coefficient of the expansion of the regular potential q_n^α in (4.12) and q_{n0}^α is a coefficient of the expansion of the singular potential q_{-n-1}^α in (4.13).

The result (5.9) relies on a specific choice of the direction of the polar axis; in order to rewrite it in a coordinate-independent form we observe that, as shown in Appendix A,

$$(ka)^n q_{n0}^{r\alpha} = \frac{a^{2n}}{n!}\left[(\mathbf{k}\cdot\nabla)^n\, q_n^\alpha(\mathbf{r})\right]_{r=0} , \tag{5.10}$$

$$(ka)^n q_{n0}^{\alpha} = \frac{a^{2n}}{n!}\left\{(\mathbf{k}\cdot\nabla)^n \left[\left(\frac{r}{a}\right)^{2n+1} q_{-n-1}^\alpha(\mathbf{r})\right]\right\}_{r=0} , \tag{5.11}$$

so that

$$\begin{aligned} L_{\mathbf{k}}^\alpha = v \sum_{n=0}^{\infty} \frac{(-a^2)^n}{n!} & \left\{(-i\mathbf{k}\cdot\nabla)^n \left[-\frac{3 j_{n+1}(ka)}{(ka)^{n+1}}\left(q_n^\alpha + \left(\frac{r}{a}\right)^{2n+1} q_{-n-1}^\alpha\right)\right.\right. \\ & \left.\left. +(2n+1)\frac{3 j_n(ka)}{(ka)^{n+2}}\left(\frac{r}{a}\right)^{2n+1} q_{-n-1}^\alpha\right]\right\}_{r=0} . \end{aligned} \tag{5.12}$$

5.2. Homogeneous integrals

In order to parallel the previous derivation we consider the equation

$$\nabla^2 q_* = q . \tag{5.13}$$

It can readily be shown that, if q is given by (4.10), a particular solution of this equation is

$$q_*(\mathbf{y}^\alpha + \mathbf{r}) = \sum_{n=-\infty}^{\infty} \frac{r^2}{4n+6} q_n^\alpha(\mathbf{r}) , \tag{5.14}$$

plus a harmonic function which, as will be clear shortly, can be assumed to be zero without loss of generality. Upon writing the integrand in (2.10) as $\nabla^2 q_*$, and applying the divergence theorem as before, we are left with

$$Q_0 = -\sum_{\alpha=1}^{N}\int_{r=a} \mathrm{d}S^\alpha \mathbf{n}\cdot\nabla q_*(\mathbf{y}^\alpha + \mathbf{r}) , \tag{5.15}$$

where the gradient is with respect to the variable $\mathbf{r}$; clearly, by the divergence theorem, the undetermined harmonic function that could be added to q_* would make no contribution to this integral. Upon substitution of (5.14) and use of (5.6) one readily finds

$$Q_0 = -v\sum_{\alpha=1}^{N}\left[q_0^\alpha - \frac{3}{2}\frac{r}{a} q_{-1}^\alpha\right]_{r=0} . \tag{5.16}$$

Note that it is not possible to find this result simply by letting $k \to 0$ in (5.12) as, for $z \to 0$, $j_n(z)/z^n \to 1/(2n+1)!!$ so that the second term in (5.12) is singular for $n = 0, 1$.

5.3. Final result

The previous results can be written in the form suggested by (3.12). We start by defining

$$
\begin{aligned}
S_n(-k^2a^2) &= \frac{3j_n(ka)}{(ka)^n} \\
&= \frac{3}{(2n+1)!!}\left[1 - \frac{(ka)^2}{1!2(2n+3)} + \frac{(ka)^4}{2!2^2(2n+3)(2n+5)} + \cdots\right] .
\end{aligned} \tag{5.17}
$$

The notation explicitly indicates that a power series expansion of S_n only contains even powers of ka, which enables us to introduce the formal differential operator $\mathcal{S}_n(a^2\nabla^2)$. The first few terms of the lower-order $\mathcal{S}_n$'s are:

$$
\mathcal{S}_0(a^2\nabla^2) = 3\left(1 + \frac{a^2}{3!}\nabla^2 + \frac{a^4}{5!}\nabla^4 + \cdots\right) , \tag{5.18}
$$

$$
\mathcal{S}_1(a^2\nabla^2) = 1 + \frac{a^2}{10}\nabla^2 + \frac{a^4}{280}\nabla^4 + \cdots , \tag{5.19}
$$

$$
\mathcal{S}_2(a^2\nabla^2) = \frac{1}{5}\left(1 + \frac{a^2}{14}\nabla^2 + \frac{a^4}{504}\nabla^4 + \cdots\right) , \tag{5.20}
$$

$$
\mathcal{S}_3(a^2\nabla^2) = \frac{1}{35}\left(1 + \frac{a^2}{18}\nabla^2 + \frac{a^4}{792}\nabla^4 + \cdots\right) . \tag{5.21}
$$

With this definition, as shown in Appendix A, it can be verified that (3.12) becomes

$$
\begin{aligned}
\beta_L \langle q\rangle(\mathbf{x}) &= -\mathcal{S}_1(a^2\nabla^2)\,(nv\overline{q_0}) \\
&\quad - \sum_{l=1}^{\infty} \frac{(-a^2)^l}{l!}\mathcal{S}_{l+1}(a^2\nabla^2)\boldsymbol{\nabla}^{(l)} \cdot \left\{ nv\overline{\left[\boldsymbol{\nabla}^{(l)}\left(q_l + \left(\frac{r}{a}\right)^{2l+1} q_{-l-1}\right)\right]}_{r=0}\right\} \\
&\quad - \frac{1}{a}\left(a^2\nabla^2\right)^{-1} \sum_{l=1}^{\infty} \frac{(-1)^l}{l!}(2l+1)\mathcal{S}_l(a^2\nabla^2)\boldsymbol{\nabla}^{(l)} \cdot \left\{ nv\overline{\left[\boldsymbol{\nabla}^{(l)}\left(r^{2l+1}q_{-l-1}\right)\right]}_{r=0}\right\} .
\end{aligned} \tag{5.22}
$$

In writing this relation we have used the fact that, noted after Equations (4.17) and (4.18), that $q^{\alpha}_{-1} = 0$.

The final step in calculating the Fourier expansion of the pressure p consists in evaluating the ensemble average of P_∞ defined in (4.9). As shown in Appendix B the result is

$$
\langle P_\infty\rangle = \mathbf{x}\cdot\left(\mathbf{G} + \frac{1}{\mathcal{V}}\int \mathrm{d}^3x\; n(\mathbf{x})\,\overline{\mathbf{F}}(\mathbf{x})\right) . \tag{5.23}
$$

The physical nature of this result becomes clearer considering the special case in which liquid and particles are subjected only to gravity. In that case $\mathbf{G} = \rho\mathbf{g}$, while $\overline{\mathbf{F}} = \mathbf{F} = v\,(\rho_D - \rho)\,\mathbf{g}$, with ρ_D the disperse-phase density and we have

$$
\mathbf{G} + \frac{1}{\mathcal{V}}\int \mathrm{d}^3x\; n(\mathbf{x})\,\overline{\mathbf{F}}(\mathbf{x}) = \left[\rho + \frac{Nv}{\mathcal{V}}(\rho_D - \rho)\right]\mathbf{g} = \frac{\rho\,(\mathcal{V} - Nv) + \rho_D Nv}{\mathcal{V}}\mathbf{g} , \tag{5.24}
$$

i.e., the total suspension weight per unit volume. The term $\langle P_\infty\rangle$ is thus an essentially hydrostatic contribution to the pressure field and is irrelevant for the analysis of the flow of the suspension.

6. The average velocity

We now turn to the calculation of the Fourier integrals (2.10) and (2.11) for the velocity. Rather than using general formulae for biharmonic functions, which can be derived as done before for harmonic functions and are presented in Appendix C, it proves more convenient to give a derivation that uses the Stokes flow equation (4.7). As before, we consider the cases $\mathbf{k} \neq 0$ and $\mathbf{k} = 0$ separately.

6.1. Non-zero wave number

As in Section 5 we start with the identity

$$\mathbf{V_k} = \int_{\mathcal{L}} \mathrm{d}^3x \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x})\, \mathbf{v}(\mathbf{x}) = -\frac{1}{k^2}\int_{\mathcal{L}} \mathrm{d}^3x \left[\nabla^2 \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x})\right] \mathbf{v}(\mathbf{x}) \;, \tag{6.1}$$

and use the divergence theorem and the Stokes equation (4.7) to rewrite the last integral as

$$\begin{aligned} \mathbf{V_k} = \;& \frac{1}{k^2}\sum_{\alpha=1}^{N} \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{y}^\alpha) \int_{r=a} \mathrm{d}S^\alpha \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}) \Bigg[-\frac{\mathrm{i}}{k^2}(\mathbf{k}\cdot\mathbf{n})\nabla q \\ & + \frac{1}{k^2}(\mathbf{n}\cdot\nabla)\nabla q + \mathrm{i}(\mathbf{k}\cdot\mathbf{n})\mathbf{v} - (\mathbf{n}\cdot\nabla)\mathbf{v}\Bigg] \;, \end{aligned} \tag{6.2}$$

where the integral over the surface of the cell, which vanishes by periodicity as before, has been omitted. Note that this expression is the sum of two terms of the form (3.1), namely

$$-\frac{1}{k^4}\sum_{\alpha=1}^{N} \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{y}^\alpha) \int_{r=a} \mathrm{d}S^\alpha \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}) \left[\mathrm{i}(\mathbf{k}\cdot\mathbf{n})\nabla q - (\mathbf{n}\cdot\nabla)\nabla q\right] \;, \tag{6.3}$$

which is identical to (5.5) with q replaced by ∇q, and

$$\frac{1}{k^2}\sum_{\alpha=1}^{N} \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{y}^\alpha) \int_{r=a} \mathrm{d}S^\alpha \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}) \left[\mathrm{i}(\mathbf{k}\cdot\mathbf{n})\mathbf{v} - (\mathbf{n}\cdot\nabla)\mathbf{v}\right] \;. \tag{6.4}$$

In this second integral, the first term can readily be evaluated by use of the boundary condition (4.15) at the particle surface. For the evaluation of the second term in (6.4) we use the representation (4.11) of $\mathbf{v}$ and the expressions (4.17) to (4.19). The method of calculation is similar to the one used before in Section 5 and we omit the details. Some useful relations are given in Appendix A. After combining all the results one finally finds

$$\begin{aligned} \mathbf{V_k} = \;& v\sum_{\alpha=1}^{N} \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{y}^\alpha) \Bigg\{-\frac{3j_1(ka)}{(ka)}\mathbf{w}_0^\alpha + \mathrm{i}a^2\frac{3j_2(ka)}{(ka)^2}(\mathbf{k}\times\boldsymbol{\Omega}^\alpha + \boldsymbol{\gamma}\cdot\mathbf{k}) \\ & + \sum_{n=1}^{\infty}\Bigg[-\frac{\mathrm{i}^n}{a(n-1)!}(2n+1)\frac{3j_n(ka)}{(ka)^{n+2}}\mathbf{k}\times\left[(\mathbf{k}\cdot\nabla)^{n-1}\nabla(r^{2n+1}\chi^\alpha_{-n-1})\right] \\ & - \frac{\mathrm{i}^{n-1}}{an!}(2n+1)(2n+3)\frac{3j_{n+1}(ka)}{(ka)^{n+3}}(k^2\mathbf{I}-\mathbf{kk})\cdot\left[(\mathbf{k}\cdot\nabla)^{n-1}\nabla(r^{2n+1}\phi^{*\alpha}_{-n-1})\right] \\ & + \frac{a\mathrm{i}^{n-1}}{n!}\frac{3j_{n-1}(ka)}{(ka)^{n+3}}(k^2\mathbf{I}-\mathbf{kk})\cdot\left[(\mathbf{k}\cdot\nabla)^{n-1}\nabla(r^{2n+1}q^\alpha_{-n-1})\right]\Bigg]\Bigg\} \end{aligned} \tag{6.5}$$

where, as in [8],

$$\phi^{*\alpha}_{-n-1} = \phi^{\alpha}_{-n-1} - \frac{a^2}{2(2n+1)} q^{\alpha}_{-n-1} \,. \tag{6.6}$$

It is understood that all the **r**-dependent terms in (6.5) are differentiated the indicated number of times and then evaluated at $\mathbf{r} = 0$.

6.2. Homogeneous part

As in Section 5 we define $\mathbf{v}_*$ such that

$$\nabla^2 \mathbf{v}_* = \mathbf{v} \,, \tag{6.7}$$

and apply the divergence theorem to the integral

$$\mathbf{V}_0 = \int_{\mathcal{L}} \mathrm{d}^3 x \mathbf{v}(\mathbf{x}) \,, \tag{6.8}$$

to find

$$\mathbf{V}_0 = -\sum_{\alpha=1}^{N} \int_{r=a} \mathrm{d}S^{\alpha} (\mathbf{n} \cdot \boldsymbol{\nabla}) \mathbf{v}_* \,. \tag{6.9}$$

The usual periodicity argument has been invoked to remove the integral over the surface of the cell. It can be shown from Lamb's solution (4.11) for **v** that

$$\begin{aligned} \mathbf{v}_*(\mathbf{y}^{\alpha} + \mathbf{r}) = \sum_{n=-\infty}^{\infty} \frac{1}{2n+3} \Bigg\{ & \frac{r^2}{8(2n+5)(n+1)} \left[(n+5) r^2 \boldsymbol{\nabla} q_n^{\alpha} - 4n \mathbf{r} q_n^{\alpha} \right] \\ & + \frac{1}{2} r^2 \boldsymbol{\nabla} \times (\mathbf{r} \chi_n^{\alpha}) + \frac{1}{n+1} \left[\frac{n+3}{2} r^2 \boldsymbol{\nabla} \phi_n^{\alpha} - n \mathbf{r} \phi_n^{\alpha} \right] \Bigg\} \,, \end{aligned} \tag{6.10}$$

plus an incompressible harmonic vector that can be assumed to be zero without loss of generality. Substituting this result in the expression (6.9) for $\mathbf{V}_0$ and proceeding as before, we find that most terms drop out because of the orthogonality of surface harmonics. One is left with

$$\mathbf{V}_0 = -v \sum_{\alpha=1}^{N} \left\{ \frac{a^2}{10} \boldsymbol{\nabla} \left[q_1^{\alpha} + 10 \left(\frac{r}{a} \right)^3 q_{-2}^{\alpha} \right] + \boldsymbol{\nabla} \left[\phi_1^{\alpha} - 2 \left(\frac{r}{a} \right)^3 \phi_{-2}^{\alpha} \right] \right\}_{r=0} \,, \tag{6.11}$$

that checks with the expression of [8]. This result can be considerably simplified by use of (4.17) and (4.18) to account for the no-slip condition at the surface of the sphere, with which it reduces to:

$$\mathbf{V}_0 = -v \sum_{\alpha=1}^{N} \mathbf{w}_0^{\alpha} \,. \tag{6.12}$$

6.3. Final result

The final result for the liquid velocity can be expressed in terms of particle averages as described in Section 3 similarly to what was done at the end of Section 5; it is:

$$\begin{aligned}\beta_L\langle\mathbf{v}\rangle = & -\mathcal{S}_1(a^2\nabla^2)\,(nv\overline{\mathbf{w}}_0) - a^2\mathcal{S}_2(a^2\nabla^2)\left[\nabla\times\left(nv\overline{\boldsymbol{\Omega}}\right) - \boldsymbol{\gamma}\cdot\nabla(nv)\right] \\ & + \left(-\nabla^2\right)^{-2}\left(\mathbf{I}\nabla^2 - \nabla\nabla\right)\cdot\sum_{l=1}^{\infty}\frac{(-1)^l}{a^3 l!}\mathcal{S}_{l-1}(a^2\nabla^2)\nabla^{(l-1)}\cdot\left(nv\overline{\nabla^{(l)}r^{2l+1}q_{-l-1}}\right) \\ & - \left(-\nabla^2\right)^{-1}\left(\mathbf{I}\nabla^2 - \nabla\nabla\right)\cdot\sum_{l=1}^{\infty}\frac{(-1)^l}{a^3 l!}(2l+1)(2l+3)\mathcal{S}_{l+1}(a^2\nabla^2)\nabla^{(l-1)} \\ & \cdot\left(nv\overline{\nabla^{(l)}r^{2l+1}\phi^*_{-l-1}}\right) \qquad (6.13) \\ & - \left(-\nabla^2\right)^{-1}\sum_{l=1}^{\infty}\frac{(-1)^l}{a^3(l-1)!}(2l+1)\mathcal{S}_l(a^2\nabla^2)\nabla\times\left[\nabla^{(l-1)}\cdot\left(nv\overline{\nabla^{(l)}r^{2l+1}\chi_{-l-1}}\right)\right] ,\end{aligned}$$

where, once again, the $\mathbf{r}$-derivatives in the particle averages are evaluated at $\mathbf{r} = 0$. Since the first term in the expression (4.1) for the continuous-phase velocity is deterministic, it remains unchanged upon averaging, so that the phase-ensemble average of the continuous-phase velocity $\mathbf{u}$ is given by

$$\langle\mathbf{u}\rangle = \mathbf{U}_\infty + \langle\mathbf{v}\rangle \ . \tag{6.14}$$

7. Disperse-phase velocity

For the considerations of the following section it is important to point out a connection between $\langle\mathbf{u}\rangle$, the phase-ensemble average of the continuous-phase velocity, and the corresponding quantity $\langle\mathbf{u}_D\rangle$ for the disperse phase. Note that, analogously to $\langle\mathbf{u}\rangle$, $\langle\mathbf{u}_D\rangle$ represents the average velocity field of the *particle material*, and it is therefore distinct, in principle, from the average velocity of the particle centers $\overline{\mathbf{w}}$, as will be seen shortly.

The phase-ensemble average of the disperse-phase velocity $\mathbf{u}_D$ is defined by a relation similar to (2.4), namely

$$\beta_D\langle\mathbf{u}_D\rangle(\mathbf{x}) = \frac{1}{N!}\int \mathrm{d}\mathcal{C}^N P(N)\chi(\mathbf{x};N)\mathbf{u}_D(\mathbf{x};N) \ . \tag{7.1}$$

Here β_D is the disperse-phase volume fraction given by

$$\beta_D(\mathbf{x}) = \frac{1}{N!}\int \mathrm{d}\mathcal{C}^N P(N)\chi(\mathbf{x};N) \ ; \tag{7.2}$$

it is immediate to verify from (2.1) that $\beta_D + \beta_L = 1$; it can also be shown (see Appendix B) that

$$\beta_D = \mathcal{S}_1(a^2\nabla^2)\,(nv) \ . \tag{7.3}$$

The first term in the expansion of $\mathcal{S}_1$ is just 1 and therefore, for a spatially uniform suspension for which all derivatives vanish, $\beta_D = nv$. This relation is often assumed in the literature but, as (7.3) shows, is an approximation only justified as a/L (where L is the macroscopic length scale giving the order of magnitude of the derivatives of the averages) becomes small.

Since, as noted earlier after (4.16), the absolute velocity of particles in different cells is different, $\langle\mathbf{u}_D\rangle$ is not periodic, while $\langle\mathbf{u}_D\rangle - \mathbf{U}_\infty$ is. Hence we expand $\langle\mathbf{u}_D\rangle - \mathbf{U}_\infty$ in a Fourier series:

$$\beta_D(\mathbf{x})\,(\langle\mathbf{u}_D\rangle - \mathbf{U}_\infty) = \mathbf{u}_0 + \sum_{\mathbf{k}\neq 0} \mathbf{u}_\mathbf{k} \exp(-\mathrm{i}\mathbf{k}\cdot\mathbf{x}) \ , \tag{7.4}$$

where

$$\mathbf{u}_0 = \frac{1}{\mathcal{V}}\int \mathrm{d}^3x \beta_D(\mathbf{x})(\langle\mathbf{u}_D\rangle - \mathbf{U}_\infty) \ , \quad \mathbf{u}_\mathbf{k} = \frac{1}{\mathcal{V}}\int \mathrm{d}^3x \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x})\, \beta_D(\mathbf{x})(\langle\mathbf{u}_D\rangle - \mathbf{U}_\infty) \ , \tag{7.5}$$

or, from (7.1),

$$\mathbf{u}_0 = \frac{1}{N!}\int \mathrm{d}\mathcal{C}^N P(N)\left[\frac{1}{\mathcal{V}}\int \mathrm{d}^3x\,\chi\,(\mathbf{u}_D - \mathbf{U}_\infty)\right] \ , \tag{7.6}$$

$$\mathbf{u}_\mathbf{k} = \frac{1}{N!}\int \mathrm{d}\mathcal{C}^N P(N)\left[\frac{1}{\mathcal{V}}\int \mathrm{d}^3x\,\chi \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x})\,(\mathbf{u}_D - \mathbf{U}_\infty)\right] \ . \tag{7.7}$$

The presence of the characteristic function χ restricts the spatial integration domain in (7.6), (7.7) to the interior of the particles. With the assumption of rigid particles we have

$$\mathbf{u}_D(\mathbf{y}^\alpha + \mathbf{r}; N) - \mathbf{U}_\infty(\mathbf{y}^\alpha + \mathbf{r}) = \mathbf{w}_0^\alpha + \boldsymbol{\Omega}^\alpha \times \mathbf{r} - \boldsymbol{\gamma}\cdot\mathbf{r} \ , \tag{7.8}$$

which enables one to calculate the spatial integrals in closed form. After manipulating the result similarly to what was done before for q and $\mathbf{v}$ we find

$$\beta_D\langle\mathbf{u}_D\rangle = \mathcal{S}_1(a^2\nabla^2)\,(nv\overline{\mathbf{w}}) + a^2\mathcal{S}_2(a^2\nabla^2)\left[\boldsymbol{\nabla}\times\left(nv\overline{\boldsymbol{\Omega}}\right) - \boldsymbol{\gamma}\cdot\boldsymbol{\nabla}(vn)\right] \ . \tag{7.9}$$

For a spatially uniform suspension this relation reduces to $\langle\mathbf{u}_D\rangle = \overline{\mathbf{w}}$, which is often assumed in the multiphase flow literature. The general result (7.9) shows, however, that, in general, the ensemble average velocity of the particle material differs from the average velocity of the particle centers $\overline{\mathbf{w}}$.

8. Mixture velocity and pressure

The previous results can be connected to related ones obtained in a completely different way in some recent papers [20, 16, 17]. It is important to establish this connection because those results heavily relied on a particular form of the ensemble-averaged momentum equations that, although motivated by a series of considerations, might still seem open to question. The fact that identical results are found by the direct route followed here should allay any lingering doubt as to their validity. In addition, it will be seen in Section 10 that the present results lead to very interesting conclusions for the structure of the viscous stress in a spatially non-uniform suspension.

In the first place, the first two terms of (6.13) for $\beta_L\langle\mathbf{v}\rangle$ are recognized to be nothing other than the average velocity of the particle material $\langle\mathbf{u}_D\rangle$ defined in Section 7 and given in (7.9). If we define the volume flux, or mixture velocity, of the suspension by

$$\mathbf{u}_m = \beta_L\langle\mathbf{u}\rangle + \beta_D\langle\mathbf{u}_D\rangle \ , \tag{8.1}$$

we find from (6.13)

$$\begin{aligned}\mathbf{u}_m = \; & \mathbf{U}_\infty + a\left(a^2\nabla^2\right)^{-2}\Bigg\{(\nabla^2\mathbf{I} - \nabla\nabla)\cdot\sum_{l=1}^{\infty}\frac{(-1)^l}{l!}\mathcal{S}_{l-1}(a^2\nabla^2)\nabla^{(l-1)} \\ & \cdot\left(nv\overline{\nabla^{(l)}(r^{2l+1}q_{-l-1})}\right) \\ & +\nabla^2(\nabla^2\mathbf{I} - \nabla\nabla)\cdot\sum_{l=1}^{\infty}\frac{(-1)^l}{l!}(2l+1)(2l+3)\mathcal{S}_{l+1}(a^2\nabla^2)\nabla^{(l-1)} \\ & \cdot\left(nv\overline{\nabla^{(l)}(r^{2l+1}\phi^*_{-l-1})}\right) \\ & +\nabla^2\sum_{l=1}^{\infty}\frac{(-1)^l}{(l-1)!}(2l+1)\mathcal{S}_l(a^2\nabla^2)\nabla\times\left[\nabla^{(l-1)}\cdot\left(nv\overline{\nabla^{(l)}(r^{2l+1}\chi_{-l-1})}\right)\right]\Bigg\}\,. \end{aligned} \tag{8.2}$$

It is readily shown from this relation that

$$\nabla\cdot\mathbf{u}_m = 0\,, \tag{8.3}$$

as expected from the fact that both phases are incompressible.

The result (8.2) may be interpreted as stating that the quantity *defined* by (8.1) is computed to have the expression given in the right-hand side of (8.2) for the present flow situation. The important feature to which we draw the reader's attention is the fact that the result for a *mixture quantity* – in this case the mixture velocity – is found to contain averages of spherical harmonics with a negative index only.

Turning now to the result for $\langle p\rangle$ given in (5.23), we find that, if we were to define

$$\begin{aligned} p_m = \; & \beta_L\langle p\rangle + \mu\mathcal{S}_1(a^2\nabla^2)\,(nv\overline{q_0}) \\ & +\mu\sum_{l=1}^{\infty}\frac{(-a^2)^l}{l!}\mathcal{S}_{l+1}(a^2\nabla^2)\nabla^{(l)}\cdot\left(nv\overline{\left[\nabla^{(l)}\left(q_l + \left(\frac{r}{a}\right)^{2l+1}q_{-l-1}\right)\right]_{r=0}}\right)\,, \end{aligned} \tag{8.4}$$

we would have from (5.23)

$$\begin{aligned} p_m = \; & \langle P_\infty\rangle - \frac{\mu}{a^3}\left(\nabla^2\right)^{-1}\sum_{\ell=1}^{\infty}\frac{(-1)^\ell}{\ell!}(2\ell+1)\mathcal{S}_\ell(a^2\nabla^2) \\ & \times\nabla^{(\ell)}\cdot\left(nv\overline{\left[\nabla^{(\ell)}\left(r^{2\ell+1}q_{-\ell-1}\right)\right]_{r=0}}\right)\,. \end{aligned} \tag{8.5}$$

The structure of this relation is similar to that of (8.2) in that the right-hand side only contains harmonics with a negative index. Just as (8.2) gives the result for the mixture quantity $\mathbf{u}_m$ defined in (8.1), (8.5) is the result for the mixture quantity p_m defined in (8.4). Here and in the following section we will justify the suggestion that this quantity is to be identified with the *mixture pressure*.

To this end we start by noting that, by using the representation (4.10) for the pressure, one readily finds

$$\frac{\mu v}{a}\nabla\left[q_1^\alpha + \left(\frac{r}{a}\right)^3 q_{-2}^\alpha\right]_{r=0} = \int_{|\mathbf{r}|=a}\mathrm{d}S_r^\alpha\, p(\mathbf{y}^\alpha+\mathbf{r})\mathbf{n} = v\left[\nabla p(\mathbf{y}^\alpha+\mathbf{r})\right]_{r=0}\,, \tag{8.6}$$

$$\frac{2\mu v}{5a}\nabla\nabla\left[q_2^\alpha + \left(\frac{r}{a}\right)^5 q_{-3}^\alpha\right]_{r=0} = \int_{|\mathbf{r}|=a} \mathrm{d}S_r^\alpha (\mathbf{nn} - \tfrac{1}{3}\mathbf{I})\, p(\mathbf{y}^\alpha + \mathbf{r}) = \tfrac{1}{5} va \left[\nabla\nabla p(\mathbf{y}^\alpha + \mathbf{r})\right]_{r=0} . \tag{8.7}$$

If we further define the mean pressure on the surface of the α particle by

$$p_e^\alpha = \frac{1}{4\pi a^2}\int_{|\mathbf{r}|=a} \mathrm{d}S_r^\alpha\, p(\mathbf{y}^\alpha + \mathbf{r}) , \tag{8.8}$$

and use the results (5.18) to (5.20) for $\mathcal{S}_l$, we can rewrite the first few terms of (8.4) as

$$p_m = \beta_L\langle p\rangle + \left(1 + \frac{a^2}{10}\nabla^2\right)(nv\overline{p_e}) - \frac{a^2}{5}\nabla\cdot\left(nv\overline{\nabla p}\right) + \frac{a^4}{14}\nabla\nabla : \left(nv\overline{\nabla\nabla p}\right) + O\left(\frac{a}{L}\right)^3 . \tag{8.9}$$

This expression coincides with the definition of mixture pressure proposed in [20], where it was written in the following equivalent form:

$$p_m = \beta_L\langle p\rangle + \left(1 + \frac{a^2}{10}\nabla^2\right)\left(\overline{\frac{nv}{4\pi a^2}\int_{|\mathbf{r}|=a} \mathrm{d}S_r\, p}\right) + \frac{a^3}{5}\nabla\cdot\left(\overline{n\int_{|\mathbf{r}|=a} \mathrm{d}S_r (-\mathbf{n})p}\right) + \frac{a^3}{14}\nabla\nabla : \left[\overline{n\int_{|\mathbf{r}|=a} \mathrm{d}S_r \left(\mathbf{nn} - \frac{1}{3}\mathbf{I}\right) p}\right] + O\left(\frac{a}{L}\right)^3 . \tag{8.10}$$

This expression was derived by identifying the component of the average stress in the suspension having the same transformation property as the pressure of an incompressible fluid upon the gauge transformation $p \to p + \psi$, with ψ a generic deterministic harmonic function. With the sole assumption of incompressibility, the calculation was general for arbitrary Reynolds numbers and rheological properties of the suspending phase (also non-Newtonian), but was based on a perturbation expansion of the averaged equations, which explains the presence of the error term in (a/L). Thus the suggestion put forward in [20] that (8.10) is to be identified with the mixture pressure is supported by the present direct derivation.

9. Discussion: the mixture pressure

The proper definition of the mixture pressure in an incompressible suspension encounters some conceptual difficulties as repeatedly noted in the literature (see, *e.g.*, [31–35]). For example, consider a disperse two-phase flow consisting of fluid spheres suspended in a continuous phase. The obvious choice for a quantity to be identified with the average mixture pressure would be the weighted sum of the pressures in the two phases:

$$p_m = \beta_L\langle p\rangle + \beta_D\langle p_D\rangle , \tag{9.1}$$

with $\langle p_D\rangle$ the phase ensemble pressure in the disperse phase. Indeed, this expression would result from the application of most formal averaging methods. But let the disperse phase become more and more viscous (*e.g.*, by decreasing the temperature). As long as it remained a fluid – however viscous – its internal pressure p_D would be well defined and this definition of average pressure would be meaningful. However, when the viscosity is large enough, the behavior of the drops would be indistinguishable from that of rigid particles and yet, although the average flow would be exactly the same in the two cases, the concept of ‘pressure’ inside a rigid particle would be devoid of physical meaning. A similar paradox arises on considering the averaged momentum equation for the disperse phase: as long as this phase consists of a fluid, most averaging methods would lead to a term involving the pressure gradient of the disperse phase but, if the disperse phase were to become more and more viscous, one would encounter the same difficulty as before.

For these reasons, several authors avoid the introduction of a disperse-phase pressure and replace it by an 'interfacial pressure', related to the mean continuous-phase pressure on the surface of the particles (see, *e.g.*, [36–39]). In the present case of spherical particles this interfacial pressure is identical with $\overline{p_e}$, the particle average of the mean surface pressure defined in (8.8). In these analyses the pressure term in the continuous-phase average momentum equation is written as (see, *e.g.*, [37, pp. 132–135], [23]):

$$\beta_L \nabla \langle p \rangle + (\overline{p_e} - \langle p \rangle)\, \nabla \beta_D \ . \qquad (9.2)$$

To see how this procedure relates to the present definition of mean mixture pressure, let us rewrite the expression (8.4) or (8.10) for p_m for the special case of a spatially uniform suspension, for which $nv = \beta_D$ (see Appendix B):

$$p_m = \beta_L \langle p \rangle + \beta_D \left(\overline{\frac{1}{4\pi a^2} \int_{|\mathbf{r}|=a} \mathrm{d}S_r\, p} \right) \ . \qquad (9.3)$$

Upon comparing with (9.1) and setting $\langle p_D \rangle = \overline{p_e}$, we obtain a well-defined entity even when the disperse phase consists of rigid particles. This identification is supported by the analysis of [33] for a uniform suspension of slightly elastic particles which explicitly shows that the isotropic part of the disperse-phase contribution to the stress tensor (which is to be identified with $\langle p_D \rangle$) is precisely equal to $\overline{p}_e$. Furthermore, upon taking the gradient of (9.3), we have

$$\nabla p_m = \beta_L \nabla \langle p \rangle + (\overline{p_e} - \langle p \rangle)\, \nabla \beta_D + \beta_D \nabla \overline{p_e} \ . \qquad (9.4)$$

The first two terms in the right-hand side are the same as those in (9.2); that the last term is necessary for consistency was shown in [31] on the basis of a less rigorous argument than that leading to the more precise result (8.4).

The above considerations ignore the differentiated terms in (8.4). The use of (9.3) in place of (8.4) is similar to approximating β_D by nv and $\langle \mathbf{u}_D \rangle$ by $\overline{\mathbf{w}}$. The omitted terms are corrections of progressively higher order in the ratio a/L of the particle radius to the macroscopic length L and may therefore be small in many cases. On the other hand, near a sedimenting front or at the interface of a bubble in a fluidized bed the local length scale is not necessarily small compared with the particle radius and these terms would be important, in particular providing a regularization of the small wavelengths.

10. Discussion: the viscous stress

From the previous results it is also possible to derive rather directly an expression for the viscous stress in the mixture. While this quantity has been well analyzed in the spatially uniform case (see, *e.g.*, [40]), no exact results are available in the presence of spatial non-uniformities.

Upon introducing the continuous-phase stress $\boldsymbol{\sigma} = -p\mathbf{I} + 2\mu\mathbf{e}$, in which $\mathbf{e}$ is the rate of deformation tensor, and ignoring for the present purposes the external force on the continuous phase,[5] the Stokes flow equation (4.8) can be averaged and rearranged in the form

[5] If the external force is conservative, its potential can simply be absorbed in the pressure and accounted for in this way, as is well known.

$$\nabla \cdot \left[-\beta_L \langle p \rangle \mathbf{I} + \mu \left(\nabla \mathbf{u}_m + (\nabla \mathbf{u}_m)^T\right)\right] = \left[\nabla \cdot (\beta_L \langle \boldsymbol{\sigma} \rangle) - \beta_L \langle \nabla \cdot \boldsymbol{\sigma} \rangle\right] \ , \tag{10.1}$$

in which the result $\beta_L \langle \boldsymbol{\sigma} \rangle = -\beta_L \langle p \rangle \mathbf{I} + \mu \left(\nabla \mathbf{u}_m + (\nabla \mathbf{u}_m)^T\right)$ exactly valid for rigid particles has been used [41]. Since, from (8.3), the mixture velocity is incompressible, the previous relation may be rewritten as

$$-\nabla p_m + \mu \nabla^2 \mathbf{u}_m = \nabla \left(\beta_L \langle p \rangle - p_m\right) + \left[\nabla \cdot (\beta_L \langle \boldsymbol{\sigma} \rangle) - \beta_L \langle \nabla \cdot \boldsymbol{\sigma} \rangle\right] \ . \tag{10.2}$$

If it were possible to describe the suspension simply by means of a mixture pressure and mixture velocity, the left-hand side would vanish and, together with the continuity equation (8.3), this would constitute a homogeneous-model formulation of the flow problem. The right-hand side then represents the effect of the discrete nature of the system, which must be representable in terms of a distributed body force and the divergence of a suitable ‘particle’ stress. Since we have at our disposal exact results for the quantities in the left-hand side, we can explicitly calculate these quantities.

By substituting (8.10) for the mixture pressure and (8.2) for the mixture velocity, we easily find the following result:

$$-\nabla p_m + \mu \nabla^2 \mathbf{u}_m = -\mathcal{S}_1(a^2\nabla^2)\left(n\overline{\mathbf{F}}\right) - \nabla \cdot \mathsf{S} - \nabla \times \left[\mathbf{U}^\chi - \nabla \times \left(\mathbf{U}^\phi + \mathbf{U}^q\right)\right] \ , \tag{10.3}$$

where we have used the fact that ([30, p. 88])

$$\mathbf{F}^\alpha = 4\pi\mu \left[\nabla \left(r^3 q^\alpha_{-2}\right)\right]_{r=a} \ , \tag{10.4}$$

and

$$\mathsf{S} = \frac{4}{3}\pi\mu \sum_{\ell=2}^{\infty} \frac{(-1)^{\ell+1}}{\ell!}(2\ell+1)\mathcal{S}_\ell(a^2\nabla^2)\nabla^{(\ell-2)} \cdot \left(n\overline{\left[\nabla^{(\ell)}\left(r^{2\ell+1}q_{-\ell-1}\right)\right]_{r=a}}\right) \ , \tag{10.5}$$

$$\begin{aligned} \mathbf{U}^\phi = \ & \frac{4}{3}\pi\mu \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1}}{\ell!}(2\ell+1)(2\ell+3)\mathcal{S}_{\ell+1}(a^2\nabla^2)\nabla^{(\ell-1)} \\ & \cdot \left(n\overline{\left[\nabla^{(\ell)}\left(r^{2\ell+1}\phi^*_{-\ell-1}\right)\right]_{r=a}}\right) , \end{aligned} \tag{10.6}$$

$$\mathbf{U}^q = \frac{4}{3}\pi\mu a^2 \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1}}{\ell!}\mathcal{S}_{\ell+1}(a^2\nabla^2)\nabla^{(\ell-1)} \cdot \left(n\overline{\left[\nabla^{(\ell)}\left(r^{2\ell+1}q_{-\ell-1}\right)\right|_{r=a}}\right) \ , \tag{10.7}$$

$$\mathbf{U}^\chi = \frac{4}{3}\pi\mu \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1}}{(\ell-1)!}(2\ell+1)\mathcal{S}_\ell(a^2\nabla^2)\nabla^{(\ell-1)} \cdot \left(n\overline{\left[\nabla^{(\ell)}\left(r^{2\ell+1}\chi_{-\ell-1}\right)\right]_{r=a}}\right) \ . \tag{10.8}$$

The term S is a symmetric second-order tensor and, since $r^{2\ell+1}q_{-\ell-1}$ is harmonic, it is traceless. For a homogeneous suspension, all terms with ℓ higher than 2 vanish and one is left with

$$\mathsf{S}_{ij} = -\tfrac{2}{3}\pi\mu\overline{\left[\partial_i\partial_j\left(r^5 q_{-3}\right)\right]_{r=a}} \ , \tag{10.9}$$

which can readily be shown to equal the average of the stresslets s^α_{ij}

$$s^\alpha_{ij} = \int_{r=a} \mathrm{d}S^\alpha \left[\tfrac{1}{2}\left(\sigma_{ik}x_j + \sigma_{jk}x_i\right) - \tfrac{1}{3}\delta_{ij}\sigma_{\ell k}x_\ell\right] n_k \ , \tag{10.10}$$

acting on the particles (see, *e.g.*, [30] p. 88). Thus, in the case of a spatially uniform suspension, the result (10.9) reduces to that given by Batchelor [40] and, in particular, leads to the Einstein viscosity correction in the dilute limit.

The last group of terms in (10.3) is the *antisymmetric* part of the stress. For a uniform suspension $\mathbf{U}^\chi$ reduces to the $\ell = 1$ contribution,

$$\mathbf{U}^\chi = 4\pi\mu n \overline{\left[\nabla\left(r^3\chi_{-2}\right)\right]_{r=a}} = -\tfrac{1}{2} n\overline{\mathbf{T}}\,, \tag{10.11}$$

where $\mathbf{T}^\alpha$ is the viscous torque acting on the α-th particle; for inertialess motion, this term vanishes unless the particles are subjected to an external couple. In this latter case, the present result shows that the external couple is equivalent to a force $-\frac{1}{2}\nabla\times\left(n\overline{\mathbf{T}}\right)$ as observed by Batchelor [40] and Brenner [3, 4] and also found in [41] by a different method.

In the uniform case, all the contributions to the the terms $\mathbf{U}^\phi$ and $\mathbf{U}^q$ except the first one vanish as well and one is left with

$$\mathbf{U}^\phi + \mathbf{U}^q = \frac{4}{15}\pi\mu n \overline{\left[\nabla\left(r^3(15\phi^*_{-2} + a^2 q_{-2})\right)\right]_{r=a}}\,. \tag{10.12}$$

From (4.18) and (10.4) this expression may equivalently be rewritten in terms of ϕ_1 and $\overline{\mathbf{F}}$. Furthermore, in the representation (4.11) of the velocity field, the terms with $n \geq 0$ represent the field 'incident' on the particle, while those with $n < 0$ are the disturbance induced by the particle. If the incident field is evaluated at the particle center $\mathbf{r} = 0$ (*i.e.*, $\mathbf{x} = \mathbf{y}^\alpha$), the only surviving term is $\nabla\phi_1$ and, with this remark, the previous expression becomes:

$$\mathbf{U}^\phi + \mathbf{U}^q = \tfrac{4}{5}\pi\mu a^3 n \overline{\left[\mathbf{u}^{\mathrm{inc}}(\mathbf{y}^\alpha) - \mathbf{w}\right]} + \tfrac{1}{5} a^2 n\overline{\mathbf{F}}\,. \tag{10.13}$$

For clarity it is worth pointing out that, in a truly uniform suspension, all derivatives except the pressure gradient vanish and, in particular, there is no contribution from viscous stresses. The proper way in which terms like S as approximated by (10.9) and $\mathbf{U}^\chi$ as approximated by (10.11) should be interpreted is thus as embodying the leading-order contributions in the spatial gradients. The term $\nabla\times\nabla\times(\mathbf{U}^\phi + \mathbf{U}^q)$ is then clearly one order smaller and, similarly, all higher-order terms in the definitions (10.5) to (10.8) are corrections of successively higher order in a/L. A consequence of this argument is that, in general, a small non-zero antisymmetric stress exists even in the absence of external couples acting on the particles. While of order $(a/L)^2$, this term is not necessarily negligible being, for instance, of the same order of magnitude as the Faxèn contribution to the particle force.

In order to establish a connection with earlier results [41], we return to (10.13) and use the hydrodynamic force on the particle

$$-\mathbf{F}^\alpha = 6\pi\mu a\left[\mathbf{u}^{\mathrm{inc}}(\mathbf{y}^\alpha) - \mathbf{w}^\alpha + \tfrac{1}{6}a^2\nabla^2\mathbf{u}^{\mathrm{inc}}(\mathbf{y}^\alpha)\right]\,, \tag{10.14}$$

to re-express $\overline{\mathbf{F}}$ finding

$$\mathbf{U}^\phi + \mathbf{U}^q = \tfrac{2}{5}\pi\mu a^3 n \overline{\left[\mathbf{w} - \mathbf{u}^{\mathrm{inc}}(\mathbf{y}^\alpha)\right]} - \tfrac{1}{5}\pi\mu a^5 n \overline{\nabla^2\mathbf{u}^{\mathrm{inc}}(\mathbf{y}^\alpha)}\,. \tag{10.15}$$

For simplicity let us retain only terms of order a^2/L^2 in the following discussion. Then we find, from (5.19) and (10.14),

$$\begin{aligned} S_1(n\overline{\mathbf{F}}) - \nabla\times\nabla\times\left(\mathbf{U}^\phi + \mathbf{U}^q\right) &\simeq n\overline{\mathbf{F}} + \nabla^2\left[\pi\mu a^3 n\overline{\left(\mathbf{w} - \mathbf{u}^{\mathrm{inc}}(\mathbf{y}^\alpha)\right)}\right] \\ &\quad -\nabla\nabla\cdot\left[\tfrac{2}{5}\pi\mu a^3 n\overline{\left(\mathbf{w} - \mathbf{u}^{\mathrm{inc}}(\mathbf{y}^\alpha)\right)}\right] \end{aligned} \tag{10.16}$$

The presence of the second term in the right-hand side was first noted in [41] in the dilute limit.

It is interesting to consider further the force term in (10.3). It is shown in Appendix B that

$$\mathcal{S}_1(a^2\nabla^2)\left(n\overline{\mathbf{F}}\right) = \frac{1}{v}\int_{|\mathbf{r}|\le a} \mathrm{d}^3r\; n(\mathbf{x}+\mathbf{r})\,\overline{\mathbf{F}}(\mathbf{x}+\mathbf{r})\;. \tag{10.17}$$

Again in the special case of a uniform suspension this term simply reduces to $n\overline{\mathbf{F}}$; in the general case, it represents the force per unit volume acting on the mixture due to the external force on the particles.[6] The difference between $\mathcal{S}_1(n\overline{\mathbf{F}})$ and $n\overline{\mathbf{F}}$ is similar to that between $\beta_D = \mathcal{S}_1(nv)$ and nv mentioned earlier. It will also be noted that, in (10.6) and (10.7), the $\ell = 1$ terms are all preceded by the operator $\mathcal{S}_1$ which, just as in (10.17), has the effect of averaging the corresponding quantities over a volume v.

In summary, the previous results show that the momentum balance in the mixture may be expressed as

$$-\nabla p_m + \nabla\cdot\mathbf{\Sigma} + \frac{1}{v}\int_{|\mathbf{r}|\le a}\mathrm{d}^3r n(\mathbf{x}+\mathbf{r})\overline{\mathbf{F}}(\mathbf{x}+\mathbf{r}) = 0\;, \tag{10.18}$$

where the total viscous stress is given by

$$\mathbf{\Sigma} = \mu\left[\nabla\mathbf{u}_m + (\nabla\mathbf{u}_m)^T\right] + \mathsf{S} + \boldsymbol{\epsilon}\cdot\left[\mathbf{U}^\chi - \nabla\times(\mathbf{U}^\phi + \mathbf{U}^q)\right] \tag{10.19}$$

in which $(\boldsymbol{\epsilon})_{ijk} = \epsilon_{ijk}$ is the alternating tensor.

The partition between symmetric and antisymmetric parts implicitly given in this equation is not unique; for example, upon observing that

$$\nabla\times\nabla\times\mathbf{U}^\phi = \nabla\cdot\left[(\nabla\cdot\mathbf{U}^\phi)\mathbf{I} - \nabla\mathbf{U}^\phi\right]\;, \tag{10.20}$$

the divergence of (10.19) may equivalently be expressed as the divergence of

$$\begin{aligned}\mathbf{\Sigma}' = {}& \mu\left[\nabla\mathbf{u}_m + (\nabla\mathbf{u}_m)^T\right] + \mathsf{S}\\ &+\tfrac{1}{2}\left[\nabla(\mathbf{U}^\phi + \mathbf{U}^q) + \left(\nabla(\mathbf{U}^\phi + \mathbf{U}^q)\right)^T\right] - \left[\nabla\cdot(\mathbf{U}^\phi + \mathbf{U}^q)\right]\mathbf{I}\\ &+\boldsymbol{\epsilon}\cdot\left[\mathbf{U}^\chi - \tfrac{1}{2}\nabla\times(\mathbf{U}^\phi + \mathbf{U}^q)\right]\;.\end{aligned} \tag{10.21}$$

The difference between (10.19) and (10.21) is

$$\begin{aligned}\mathbf{\Sigma}' - \mathbf{\Sigma} = {}& \tfrac{1}{2}\left[\nabla(\mathbf{U}^\phi + \mathbf{U}^q) + \left(\nabla(\mathbf{U}^\phi + \mathbf{U}^q)\right)^T\right]\\ &-\left[\nabla\cdot(\mathbf{U}^\phi + \mathbf{U}^q)\right]\mathbf{I} + \tfrac{1}{2}\boldsymbol{\epsilon}\cdot\left[\nabla\times(\mathbf{U}^\phi + \mathbf{U}^q)\right],\end{aligned} \tag{10.22}$$

which is readily seen to be divergencelss. In the form (10.19) the symmetric part of the stress acquires a new contribution (first term in the second line of 10.21) and an isotropic part, and the antisymmetric part (last line of 10.21) is also modified. It is this latter form that was proposed in our recent paper [20] by a different argument. With the hindsight afforded by the present results, the form (10.19) seems simpler and more natural.

[6] Note that the integral of this term over the cell precisely cancels the integral in Equation (5.23) so that the total net force on the system vanishes.

11. Conclusions

In the present paper we have calculated the ensemble averages of fluid pressure and velocity fields in a suspension constructed by the infinite repetition of a fundamental cell containing N randomly placed equal rigid spheres. This arrangement forms the basis for a common method for the direct numerical simulation of unbounded suspensions (see, *e.g.*, [5, 8, 9, 16, 17, 27]). The final expressions (*cf.*, Equations 10.5–10.8) are in terms of summations of multipole contributions in which progressively higher-order terms introduce corresponding higher-order spatial derivatives, which suggests an interpretation as an expansion in powers of a/L, where L is a macroscopic length scale.

The results are of interest in themselves and may be applied to other physical problems in which similar averages of quantities defined by the Laplace or biharmonic equation are involved. In this paper, the focus has been on the averaged description of a suspension: we have used the results to define the proper quantity to be identified with the mixture pressure and we have deduced a computable expression for the mixture stress. As shown in [16–18], one can use this expression to systematically derive constitutive relations for the averaged equations. We have also shown that the results found here, while consistent with those obtained by different methods in earlier work [20, 41], permit to simplify them considerably thus reducing the complexity of finding closure relations for averaged equations. In addition, the earlier method of [20] is applicable also to non-Newtonian flow and flow at finite Reynolds numbers and therefore, with the insight derived here, it will be possible to derive simpler results for these important cases as well.

In closing we note a connection between the results presented in this paper and integrals of the functions introduced by Hasimoto [28]. Hasimoto's function S_1 satisfies Equation (4.4) given before. Any harmonic function defined in the periodic cell and regular outside the N spheres can be represented as a superposition of S_1 and its derivatives (see, *e.g.*, [15]):

$$\Phi = \sum_{\alpha=1}^{N} \mathcal{H}^{\alpha} S_1(\mathbf{x} - \mathbf{y}^{\alpha}) \ , \tag{11.1}$$

where $\mathcal{H}$ is a suitable differential operator (see [15] and [8] for details). By means of the formulae given in the references, the expansion (11.1) can be expressed in terms of spherical harmonics in the neighborhood of each particle. The results of Section 5 can then be used to calculate the volume integrals of Φ over the domain $\mathcal{L}$ defined in (2.9). Without getting into details, it is clear that the results given in this paper can be used to evaluate such integrals.

By similar arguments one can use the results of Appendix C to calculate integrals of S_2, the second function defined by Hasimoto, and its derivatives. The relation between S_1 and S_2 is $\nabla^2 S_2 = S_1$. It is sufficient to note that a biharmonic function may be expanded as

$$\Psi = \sum_{\alpha=1}^{N} \left[\mathcal{G}^{\alpha} S_1(\mathbf{x} - \mathbf{y}^{\alpha}) + \mathcal{F}^{\alpha} S_2(\mathbf{x} - \mathbf{y}^{\alpha}) \right] \ , \tag{11.2}$$

where $\mathcal{G}$, $\mathcal{F}$ are differential operators similar in structure to $\mathcal{H}$.

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Appendix A

We collect here some of the details of the calculation.

In order to prove (5.10) we note that $(\mathbf{k}\cdot\nabla)^n$ is just k^n times the n-th order derivative in the direction z of the polar axis. From the normalization $P_n(1) = 1$ it is obvious that

$$r^n P_n(\cos\theta) = z^n + \cdots\ , \tag{A.1}$$

where the omitted terms involve powers of z lower than n since $r^n P_n$ is a homogeneous polynomial of degree n in x, y, z. Similarly, for $m > 0$, $r^n P_n^m \cos(m\varphi)$ and $r^n P_n^m \sin(m\varphi)$ are homogeneous polynomials in x, y, z containing z to a power less than n (see, *e.g.*, [42, p. 137]). Therefore

$$(\mathbf{k}\cdot\nabla)^n \left[r^n P_n(\cos\theta)\right] = k^n n!\ , \tag{A.2}$$

(where $k = |\mathbf{k}|$) from which (5.10) immediately follows; the analogous relation (5.11) follows by a similar argument upon noting that, *e.g.*, $r^{2n+1} q_{-n-1}$ is a solid harmonic of order n.

The following results valid for any solid harmonic ϕ_n of degree n regular at the origin can also be proven

$$\int_{r=a} \mathrm{d}S \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})\nabla\phi_n = \frac{4\pi a^{n+1}}{(n-1)!}\mathrm{i}^{n-1} j_{n-1}(ka)\,(\mathbf{m}\cdot\nabla)^{n-1}\nabla\phi_n\ , \tag{A.3}$$

$$\begin{aligned}\int_{r=a} \mathrm{d}S \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})\mathbf{r}\phi_n = \ & 3v\left[\frac{a^n}{n!}\mathbf{m}\,(\mathbf{m}\cdot\nabla)^n\,\mathrm{i}^{n+1} j_{n+1}(ka)\phi_n\right. \\ & \left. + \frac{\mathrm{i}^{n-1}a^n}{(n-1)!}(\mathbf{m}\cdot\nabla)^n\,\mathrm{i}^{n-1}\frac{j_n(ka)}{ka}\nabla\phi_n\right]\ ,\end{aligned} \tag{A.4}$$

$$\int_{r=a} \mathrm{d}S \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})\,(\nabla\phi_n)\times\mathbf{r} = \frac{4\pi a^{n+2}}{(n-1)!}\mathrm{i}^n j_n(ka)\mathbf{m}\times\left[(\mathbf{m}\cdot\nabla)^{n-1}\nabla\phi_n\right]\ , \tag{A.5}$$

where $\mathbf{m} = \mathbf{k}/k$. These expressions are useful to evaluate the surface integrals that arise in the calculation of the velocity.

A more detailed justification of the step leading from (5.12) to (5.23) or, analogously, from (6.5) to (6.13) is as follows. For brevity, define the following ℓ-th order tensors pertaining to particle α:

$$\mathcal{A}_\ell^\alpha = -v\frac{(-a^2)^\ell}{\ell!}\nabla^{(\ell)}\left[q_\ell + \left(\frac{r}{a}\right)^{2\ell+1} q_{-\ell-1}\right]_{r=0}\ , \tag{A.6}$$

$$\mathcal{B}_\ell^\alpha = v\frac{(-a^2)^\ell}{\ell!}(2\ell+1)\nabla^{(\ell)}\left[\left(\frac{r}{a}\right)^{2\ell+1} q_{-\ell-1}\right]_{r=0}\ , \tag{A.7}$$

where the gradient is with respect to $\mathbf{r}$. We may then write

$$\exp(-\mathrm{i}\mathbf{k}\cdot\mathbf{x})L_{\mathbf{k}}^{\alpha} = \left[\sum_{\ell=0}^{\infty} S_{\ell+1}(a^2\nabla^2)\boldsymbol{\nabla}^{(\ell)}\cdot\mathcal{A}_\ell^\alpha + (-a^2\nabla^2)^{-1}S_\ell(a^2\nabla^2)\boldsymbol{\nabla}^{(\ell)}\cdot\mathcal{B}_\ell^\alpha\right] \exp(-\mathrm{i}\mathbf{k}\cdot\mathbf{x})\ , \tag{A.8}$$

where the differential operators now act on the variable $\mathbf{x}$.

According to the definitions of Sections 2 and 3, the Fourier expansion (2.5) for q may be written as

$$\beta_L(\mathbf{x})\langle q\rangle(\mathbf{x})-q_0=\sum_{\mathbf{k}\neq 0}\left\{\frac{1}{\mathcal{V}}\int \mathrm{d}^3x'\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x}')\left[\frac{1}{N!}\int \mathrm{d}\mathcal{C}^N P(N)\sum_{\alpha=1}^{N}\delta(\mathbf{x}'-\mathbf{y}^\alpha)L_{\mathbf{k}}^\alpha\right]\right\}\exp(-\mathrm{i}\mathbf{k}\cdot\mathbf{x}), \tag{A.9}$$

which, from (A.8) and using the definition (3.4) of particle averages, on the basis of which

$$n(\mathbf{x}')\overline{\mathcal{A}_\ell}(\mathbf{x}') = \frac{1}{N!}\int \mathrm{d}\mathcal{C}^N P(N)\sum_{\alpha=1}^{N}\delta(\mathbf{x}'-\mathbf{y}^\alpha)\mathcal{A}_\ell^\alpha\ , \tag{A.10}$$

$$n(\mathbf{x}')\overline{\mathcal{B}_\ell}(\mathbf{x}') = \frac{1}{N!}\int \mathrm{d}\mathcal{C}^N P(N)\sum_{\alpha=1}^{N}\delta(\mathbf{x}'-\mathbf{y}^\alpha)\mathcal{B}_\ell^\alpha\ , \tag{A.11}$$

becomes

$$\begin{aligned}\beta_L(\mathbf{x})\langle q\rangle(\mathbf{x}) - q_0 = &\sum_{\ell=0}^{\infty}\Big\{\mathcal{S}_{\ell+1}(a^2\nabla^2)\\ &\boldsymbol{\nabla}^{(\ell)}\cdot\left[\sum_{\mathbf{k}\neq 0}\left(\frac{1}{\mathcal{V}}\int \mathrm{d}^3x'\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x}')n(\mathbf{x}')\overline{\mathcal{A}_\ell}(\mathbf{x}')\right)\exp(-\mathrm{i}\mathbf{k}\cdot\mathbf{x})\right]\\ &+(-a^2\nabla^2)^{-1}\mathcal{S}_\ell(a^2\nabla^2)\\ &\boldsymbol{\nabla}^{(\ell)}\cdot\left[\sum_{\mathbf{k}\neq 0}\left(\frac{1}{\mathcal{V}}\int \mathrm{d}^3x'\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x}')n(\mathbf{x}')\overline{\mathcal{B}_\ell}(\mathbf{x}')\right)\right]\exp(-i\mathbf{k}\cdot\mathbf{x})\Big\}\ ,\end{aligned} \tag{A.12}$$

But, clearly, the quantity in brackets are the Fourier expansions of $n(\mathbf{x})\overline{\mathcal{A}_\ell}(\mathbf{x})$ and $n(\mathbf{x})\overline{\mathcal{B}_\ell}(\mathbf{x})$ respectively, except for the term corresponding to $\mathbf{k}=0$. Because of the differential operator $\boldsymbol{\nabla}^{(\ell)}$, the contribution of this zero-mode is only important for $\ell=0$. Since, as noted before, $q_{-1}=0$, $\mathcal{B}_0^\alpha=0$ while, from (3.7), we have

$$(n\overline{\mathcal{A}_0})_0=\frac{1}{\mathcal{V}}\int \mathrm{d}^3x' n(\mathbf{x}')\overline{\mathcal{A}_0}(\mathbf{x}') = \frac{1}{\mathcal{V}}\int \mathrm{d}^3x'\frac{1}{N!}\int \mathrm{d}\mathcal{C}^N P(N)\left[\sum_{\alpha=1}^{N}\delta(\mathbf{x}'-\mathbf{y}^\alpha)\left(-vq_0^\alpha\right)\right] \tag{A.13}$$

which is identical to the contribution of Q_0. Hence, writing the $\ell=0$ term explicitly, (A.12) becomes

$$\begin{aligned}\beta_L(\mathbf{x})\langle q\rangle(\mathbf{x}) = &-\mathcal{S}_1(nv\overline{q}_0)+\sum_{\ell=1}^{\infty}\Big\{\mathcal{S}_{\ell+1}\boldsymbol{\nabla}^{(\ell)}\cdot\left[n(\mathbf{x})\overline{\mathcal{A}_\ell}(\mathbf{x})\right]\\ &+(-a^2\nabla^2)^{-1}\mathcal{S}_\ell\boldsymbol{\nabla}^{(\ell)}\cdot\left[n(\mathbf{x})\overline{\mathcal{B}_\ell}(\mathbf{x})\right]\Big\}\end{aligned} \tag{A.14}$$

which is (5.22).

Appendix B. the operator $\mathcal{S}_1$

Consider, in the periodic setting of this paper, a generic quantity given by

$$T = \int_{|\mathbf{r}|\le a} d^3 r\, n(\mathbf{x}+\mathbf{r}) f(\mathbf{x}+\mathbf{r}) \ , \tag{B.1}$$

where f is such that the integral is well defined. Upon expanding T in a Fourier series we have

$$T = \sum_{\mathbf{k}} (T)_{\mathbf{k}} \exp\left(-i\mathbf{k}\cdot\mathbf{x}\right) \ , \tag{B.2}$$

where

$$\begin{aligned}
(T)_{\mathbf{k}} &= \frac{1}{\mathcal{V}} \int d^3 x \exp\left(i\mathbf{k}\cdot\mathbf{x}\right) T \\
&= \frac{1}{\mathcal{V}} \int_{|\mathbf{r}|\le a} d^3 r \int d^3 x \exp\left(i\mathbf{k}\cdot\mathbf{x}\right) n(\mathbf{x}+\mathbf{r}) f(\mathbf{x}+\mathbf{r}) \\
&= \frac{1}{\mathcal{V}} \int_{|\mathbf{r}|\le a} d^3 r \exp\left(-i\mathbf{k}\cdot\mathbf{r}\right) \int d^3 z \exp\left(i\mathbf{k}\cdot\mathbf{z}\right) n(\mathbf{z}) f(\mathbf{z}) \\
&= (nf)_{\mathbf{k}} \int_{|\mathbf{r}|\le a} d^3 r \exp\left(-i\mathbf{k}\cdot\mathbf{r}\right) \ .
\end{aligned} \tag{B.3}$$

But

$$\int_{|\mathbf{r}|\le a} d^3 r \exp\left(-i\mathbf{k}\cdot\mathbf{r}\right) = v S_1(-k^2 a^2) \ . \tag{B.4}$$

Upon substituting into (B.2) we have

$$\begin{aligned}
T &= \sum_{\mathbf{k}} v S_1(-k^2 a^2) \exp\left(-i\mathbf{k}\cdot\mathbf{x}\right) (nf)_{\mathbf{k}} = v \mathcal{S}_1(a^2\nabla^2) \sum_{\mathbf{k}} \exp\left(-i\mathbf{k}\cdot\mathbf{x}\right) (nf)_{\mathbf{k}} \\
&= \mathcal{S}_1(a^2\nabla^2)(nvf) \ .
\end{aligned} \tag{B.5}$$

With $f = 1$, this result proves the relation (7.3) between β_D and nv while, with $f = \overline{\mathbf{F}}$, we recover (10.17).

A similar argument enables us to prove the relation (5.23). Here, from the definition (2.4) of phase average, we have

$$\beta_L \left\langle \sum_{\alpha=1}^{N} \mathbf{F}^{\alpha} \right\rangle = \frac{1}{N!} \int d\mathcal{C}^N P(N)(1-\chi) \sum_{\alpha=1}^{N} \mathbf{F}^{\alpha} \ . \tag{B.6}$$

Again we expand in a Fourier series the coefficients of which are found to be

$$\begin{aligned}
&\frac{1}{\mathcal{V}} \int d^3 x \exp\left(i\mathbf{k}\cdot\mathbf{x}\right) \frac{1}{N!} \int d\mathcal{C}^N P(N)(1-\chi) \sum_{\alpha=1}^{N} \mathbf{F}^{\alpha} \\
&\quad = \left[\frac{1}{N!} \int d\mathcal{C}^N P(N) \sum_{\alpha=1}^{N} \mathbf{F}^{\alpha} \right] \left[\frac{1}{\mathcal{V}} \int d^3 x \exp\left(i\mathbf{k}\cdot\mathbf{x}\right)(1-\chi) \right] \ .
\end{aligned} \tag{B.7}$$

But

$$\frac{1}{N!}\int \mathrm{d}\mathcal{C}^N P(N)\sum_{\alpha=1}^{N}\mathbf{F}^\alpha = \frac{1}{N!}\int \mathrm{d}\mathcal{C}^N P(N)\int \mathrm{d}^3x'\sum_{\alpha=1}^{N}\delta(\mathbf{x}'-\mathbf{y}^\alpha)\mathbf{F}^\alpha = \int \mathrm{d}^3x' n(\mathbf{x}')\overline{\mathbf{F}}(\mathbf{x}') \quad \text{(B.8)}$$

while the other factor in (B.7) is just the Fourier coefficient of β_L. The Fourier series can then be summed to give (5.23).

Appendix C. Biharmonic functions

Since the components of the liquid velocity satisfy the biharmonic equation, the calculation of Section 6 could also have been undertaken by developing first general results for biharmonic functions similarly to what was done in Section 5. It turns out that, in the particular case considered here, the route followed in Section 6 is somewhat more convenient. However, for completeness, we now present general results for functions Ψ that are biharmonic outside the spheres centered at $\mathbf{y}^\alpha$.

Just as in (5.2) we rewrite the integral in question identically as

$$\Psi_{\mathbf{k}} = -\frac{1}{k^2}\int_{\mathcal{L}} \mathrm{d}\mathbf{r}\left[\nabla^2\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})\right]\Psi(\mathbf{r})\ . \quad \text{(C.1)}$$

Two applications of Green's theorem leave

$$\begin{aligned}\Psi_{\mathbf{k}} = &\ \frac{1}{k^4}\int_A \mathrm{d}A\left[\left(\nabla^2\Psi(\mathbf{r})\right)\boldsymbol{\nabla}\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}) - \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})\boldsymbol{\nabla}\left(\nabla^2\Psi(\mathbf{r})\right)\right]\cdot\mathbf{n} \\ &-\frac{1}{k^2}\int_A \mathrm{d}A\left[\Psi(\mathbf{r})\boldsymbol{\nabla}\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}) - \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})\boldsymbol{\nabla}\Psi(\mathbf{r})\right]\cdot\mathbf{n}\ .\end{aligned} \quad \text{(C.2)}$$

The integral over the cell surface vanishes by periodicity as before, so that

$$\begin{aligned}\Psi_{\mathbf{k}} = &\ -\frac{1}{k^4}\sum_{\alpha=1}^{N}\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{y}^\alpha)\int_{r=a}\mathrm{d}S^\alpha\mathbf{n}\cdot\left[\left(\nabla^2\Psi(\mathbf{y}^\alpha+\mathbf{r})\right)\boldsymbol{\nabla}\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})\right. \\ &\left. - \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})\boldsymbol{\nabla}\left(\nabla^2\Psi(\mathbf{y}^\alpha+\mathbf{r})\right)\right] \\ &+\frac{1}{k^2}\sum_{\alpha=1}^{N}\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{y}^\alpha)\int_{r=a}\mathrm{d}S^\alpha\mathbf{n}\cdot\left[\Psi(\mathbf{y}^\alpha+\mathbf{r})\boldsymbol{\nabla}\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})\right. \\ &\left. - \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})\boldsymbol{\nabla}\Psi(\mathbf{y}^\alpha+\mathbf{r})\right]\ .\end{aligned} \quad \text{(C.3)}$$

It is easy to show that, since Ψ is biharmonic, it must have a structure of the type

$$\Psi(\mathbf{r}^\alpha+\mathbf{r}) = \sum_{n=-\infty}^{\infty}\left[\frac{r^2}{4n+6}\psi_n^\alpha + \tilde{\psi}_n^\alpha\right]\ , \quad \text{(C.4)}$$

where ψ_n^α and $\tilde{\psi}_n^\alpha$ are both n-th order spherical harmonics. Again using (5.6), we can then show that

$$\begin{aligned}\Psi_{\mathbf{k}} = &\ \frac{1}{k^2}\sum_{\alpha=1}^{N}\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{y}^\alpha)\sum_{n=-\infty}^{\infty}\frac{1}{a}\int_{r=a}\mathrm{d}S^\alpha(\mathrm{i}ka\ \cos\theta - n)\exp(\mathrm{i}k\zeta a)\left(\tilde{\psi}_n^\alpha - \frac{1}{k^2}\psi_n^\alpha\right) \\ &+\frac{1}{k^2}\sum_{\alpha=1}^{N}\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{y}^\alpha)\sum_{n=-\infty}^{\infty}\frac{a}{4n+6}\left[k\frac{\partial}{\partial k} - (n+2)\right]\int_{r=a}\mathrm{d}S^\alpha\psi_n^\alpha\exp(\mathrm{i}k\zeta a)\ .\end{aligned} \quad \text{(C.5)}$$

Again using the expansion (5.8), we rewrite this relation as

$$\begin{aligned}\Psi_{\mathbf{k}} = \; & \frac{3v}{ka}\sum_{\alpha=1}^{N}\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}^{\alpha})\left[-\sum_{n=0}^{\infty}\mathrm{i}^{n} j_{n+1}(ka)\left(\tilde{\psi}_{n0}^{r\alpha}-\frac{1}{k^{2}}\psi_{n0}^{r\alpha}\right)\right. \\ & \left.+\sum_{n=1}^{\infty}\mathrm{i}^{n} j_{n-1}(ka)\left(\tilde{\psi}_{n0}^{\alpha}-\frac{1}{k^{2}}\psi_{n0}^{\alpha}\right)\right] \\ & +\frac{3va}{2k}\sum_{\alpha=1}^{N}\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}^{\alpha})\left[\sum_{n=0}^{\infty}\frac{\mathrm{i}^{n}}{2n+3}\psi_{n0}^{r\alpha}\left(j_{n-1}(ka)-\frac{2n+3}{ka}j_{n}(ka)\right)\right. \\ & \left.+\sum_{n=1}^{\infty}\frac{\mathrm{i}^{n}}{2n+3}\psi_{n0}^{\alpha}\left(j_{n-1}(ka)-\frac{2}{ka}j_{n}(ka)\right)\right]\,. \end{aligned} \tag{C.6}$$

For the homogeneous integrals we define Ψ^{*} such that

$$\nabla^{2}\Psi^{*}=\Psi\,, \tag{C.7}$$

apply Green's theorem, and drop the contribution of the cell surface as before to find:

$$\Psi_{0}=-\sum_{\alpha=1}^{N}\int_{r=a}\mathrm{d}S^{\alpha}\mathbf{n}\cdot\nabla\Psi^{*}(\mathbf{r}^{\alpha}+\mathbf{r})\,. \tag{C.8}$$

By using the representation (C.4) of Ψ and the relation

$$\nabla^{2}(r^{4}\psi_{n}^{\alpha})=4(2n+5)r^{2}\psi_{n}^{\alpha}\,, \tag{C.9}$$

and proceeding as for the derivation of (5.14), we find

$$\Psi^{*}=\sum_{n=-\infty}^{\infty}\frac{r^{2}}{4n+6}\left[\frac{r^{2}}{8n+20}\psi_{n}^{\alpha}+\tilde{\psi}_{n}^{\alpha}\right]\,, \tag{C.10}$$

plus a harmonic function which, as before, can be taken to be zero. With this formula, we obtain

$$\Psi_{0}=-\sum_{\alpha=1}^{N}\int_{r=a}\mathrm{d}S^{\alpha}\sum_{n=-\infty}^{\infty}\frac{1}{4n+6}\left[\frac{n+4}{8n+20}r^{3}\psi_{n}^{\alpha}+(n+2)r\tilde{\psi}_{n}^{\alpha}\right]\,, \tag{C.11}$$

which, after integration, becomes

$$\Psi_{0}=-v\sum_{\alpha=1}^{N}\left(\frac{a^{2}}{10}\psi_{0}^{\alpha}+\tilde{\psi}_{0}^{\alpha}+\frac{3a^{2}}{8}\psi_{-1}^{\alpha}+\frac{3}{2}\tilde{\psi}_{-1}^{\alpha}\right)\,. \tag{C.12}$$

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